MST204.7

The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

Unit 7 Oscillations and energy

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MST204 Mathematical Models and Methods

Unit 7
Oscillations and energy

Prepared by the Course Team

The Open University

The Open University Press, Walton Hall, Milton Keynes, MK7 6AA.

First published 1982. New edition 1990. Reprinted 1993, 1995, 1997

Designed by the Graphic Design Group of the Open University.

Typeset by the TEXproject, Faculty of Mathematics, The Open University.

Printed in Great Britain by BPC Wheatons Ltd, Exeter

ISBN 0 7492 2031 7

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2.4

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Introduction

> After all our hearts beat, our lungs oscillate, we shiver when we are cold, we sometimes snore, we can hear and speak because our eardrums and larynges vibrate. The light waves which permit us to see entail vibration. We move by oscillating our legs. We cannot even say 'vibration' properly without the tip of the tongue oscillating ... Even the atoms of which we are constituted vibrate.
>
> *R. Davies, 1965*

Units 7 and *8* are mainly about mechanical systems which oscillate or vibrate (the two words can be used interchangeably). Figure 1 shows in diagrammatic form some of the systems we shall discuss: in each case part of the system moves up and down, or to and fro, about an average position. We shall attempt to explain these oscillations by using the laws of Newtonian mechanics to set up second-order linear differential equations which can be solved by the methods of *Unit 6*.

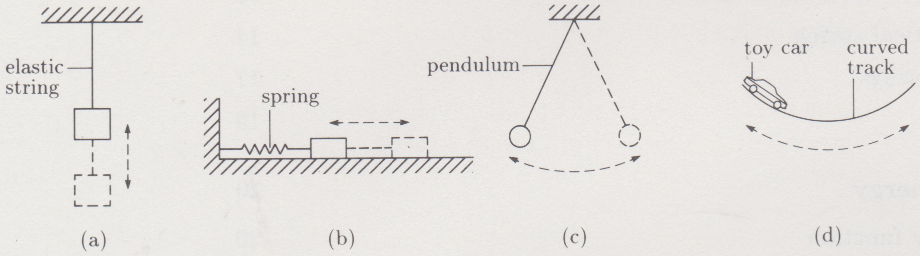

Figure 1 Examples of oscillatory motion.

The system in Figure 1(a) is a weight hanging on an elastic string; this is a simplified representation of the system for which experiments are described in Section 1. Figure 1(b) is a diagram of a particle connected to a spring which is, in turn, attached to a fixed point at its other end. This might be a model for a railway truck which has run into spring-loaded buffers. Figure 1(c) represents a pendulum such as might form part of a clock, and Figure 1(d) shows a toy car rolling backwards and forwards along a curved track; these are both systems to be seen in the television programme which accompanies this unit.

The important point is that all of these systems (as well as many others) can, subject to certain restrictions, be modelled in the same way using a mathematical model called *simple harmonic motion*, which is described by the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

where ω is a positive constant. This model is developed by using Newton's second law. The principal feature leading to this oscillatory behaviour is a force which depends linearly on position. This force law was postulated in 1678 by Newton's contemporary, Robert Hooke.

For forces which depend only on position we shall establish the important principle of the *conservation of energy* and show how this can be used as an alternative to Newton's second law for analysing suitable systems. Although in this course we derive this principle for the case when the force depends only on position, it is of much wider application. By introducing other forms of energy, such as thermal, electromagnetic, chemical, and so on, conservation of energy can be extended beyond the field of mechanics and becomes one of the fundamental laws of physics.

Study guide

This unit contains five sections, which should be studied in the order that they appear. Section 1 describes a simple experiment involving a vibrating system. Section 2 uses Newton's second law to model the oscillations of a single particle attached to springs. The law of conservation of energy is developed in Section 3 for forces which depend only on position, and is applied to systems involving gravity and springs. The television

section (Section 4) is mainly about the application of energy conservation to oscillating systems. It will be easier to follow if you have prepared in advance by reading up to the end of Subsection 4.2. Section 5 provides a set of exercises which can be used for additional practice or for revision. There is no audio-tape associated with this unit.

If you are short of time then you should concentrate your efforts on Sections 2 and 3. These contain important ideas which will be built upon in later units.

1 A home-made oscillating system

1.1 A description of the experiment

To introduce the subject of oscillations we shall describe the motion of a simple mechanical system set up at home. The system is illustrated in Figure 1; it was constructed by tying several rubber bands together to produce an elastic string which was used to suspend a bag from a door lintel. The bag itself had negligible mass, but it contained a number of identical coins.

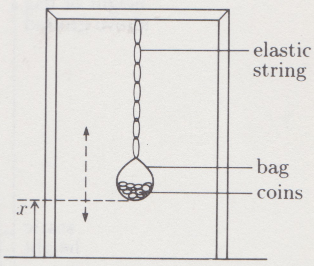

Figure 1
The home-made oscillating system.

In order to describe the position of the bag, we need to set up an x-axis. Suppose that this is chosen to point vertically upwards, with the origin at ground level. The position of the bag is then described by the variable x in Figure 1, which is the height of the bag above the ground, in metres. This height x was measured when the bag was static (i.e. permanently at rest). Measurements were also made of how x depended on time when the bag oscillated up and down. These measurements were repeated with different numbers of coins in the bag.

1.2 The experimental results

A static bag

With 5 coins placed in the bag it was found to remain static when the distance between the bag and the ground was 1.53 metres. We shall call this value of x the static height of the bag and denote it by the symbol h_5, where the suffix indicates that the bag contained 5 coins.

On adding more coins to the bag, the elastic string became increasingly stretched so that the static height was reduced. Table 1 and Figure 2 show the static heights measured when the bag contained 5, 10, 15, 20 and 25 coins. Each coin had a mass of 0.01 kg.

Table 1

Number of coins, n	5	10	15	20	25
Static height above ground, h_n (metres)	1.53	1.38	1.15	0.91	0.69

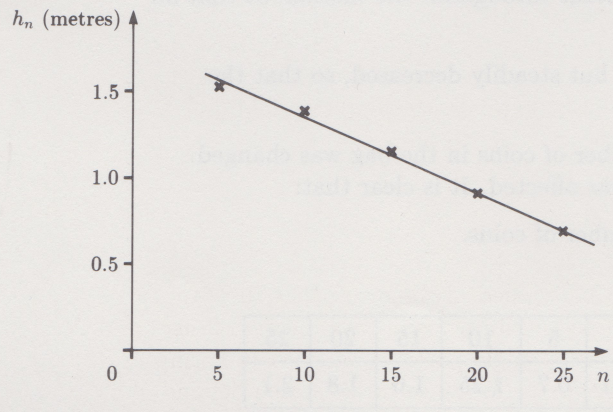

Figure 2 The static height, h_n, plotted against the number of coins, n.

Notice that the number of coins, n, and the static height, h_n, are related in an approximately linear way when the bag is between 0.7 and 1.5 metres above the ground. The straight line which best fits the experimental data points is given by

$$h_n = 1.78 - 0.0430n, \quad \text{for } 0.7 < h_n < 1.5. \tag{1}$$

An oscillating bag

To study oscillations, the bag containing coins was pulled below its static height and then released at time 0. Care was taken that the string did not go slack at any stage, and the resulting oscillations were timed. Figure 3 gives a rough idea of the sort of motion which was observed: the bag rose and fell many times before eventually settling down at its static height.

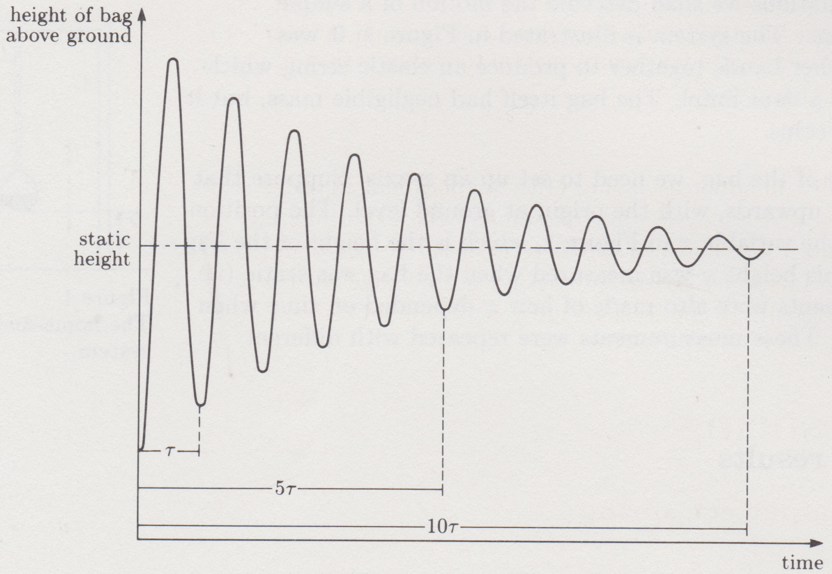

Figure 3 The observed motion of the bag of coins.

Consider, for the moment, the motion that occurs between two successive minima. As it rises from a minimum, the bag picks up speed until it passes through the static height. Then it slows down and comes momentarily to rest at the top of its flight. After that, it falls down again, picking up speed until it passes through the static height for a second time. From then on, the bag slows down until it comes momentarily to rest at the next minimum. This motion, from one minimum to the next, is known as a **cycle** of the oscillation. The time that elapses between successive minima is called the **period** of the cycle and is denoted by the symbol τ. The maximum distance by which the bag rises above, or falls below, the static height is called the **amplitude** of the cycle, and is denoted by the symbol A.

Two facts about period and amplitude were observed and are shown in Figure 3:

(i) the period remained approximately constant throughout the motion, so that all cycles took about the same time;

(ii) the amplitude did not remain constant, but steadily decreased, so that the oscillations died away.

A third fact became apparent when the number of coins in the bag was changed. Table 2 and Figure 4 show how the period was affected. It is clear that:

(iii) the period was greater for a greater number of coins.

Table 2

Number of coins, n	5	10	15	20	25
Period of oscillation, τ (seconds)	0.7	1.25	1.6	1.8	2.1

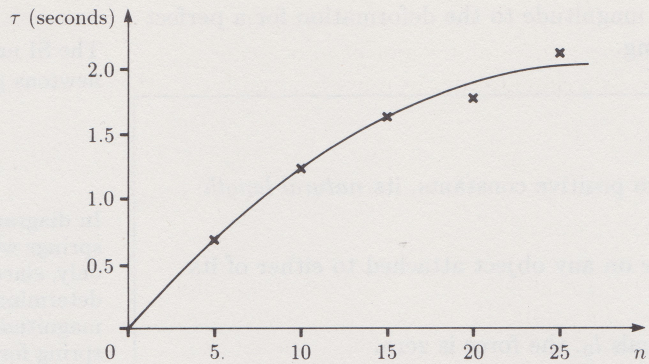

Figure 4 The period of the oscillation, τ, plotted against the number of coins, n.

Exercise 1

Figure 4 indicates that the period of oscillation, τ, does not depend linearly on the number of coins, n. By plotting τ^2 against n, show that a fair approximation to reality is that τ^2 is directly proportional to n.

[*Solution on page 34*]

At this point we shall temporarily abandon discussion of the experiment in order to set up a general mathematical model which can account for some of the experimental findings. This model also provides a link between the experimental situation and the systems shown in Figure 1 of the Introduction. In Subsection 2.4 we shall return for a second look at the experimental results in the light of the general model.

2 The perfect spring

2.1 The force law for a perfect spring

The word 'spring' suggests a helical coil made of metal wire, and in representing a 'perfect spring' diagrammatically, as in Figure 1, it is this physical realization which is portrayed. However, a 'perfect spring' is in fact nothing more than a model for a particular, simple variation of force with displacement and, while it may be used as an idealized representation of a real coil spring, it can also be made to stand for part of a system which contains no spring at all but which behaves in the appropriate way. In our idealized model the 'spring' has zero mass.

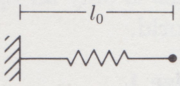

Figure 1
A spring at its natural length.

Suppose that Figure 1 represents a real spring, which is in its undisturbed state. One end is attached to a fixed object such as a wall, and the other end is free to move but does not do so because no force acts upon it. The length of the spring under these circumstances is its **natural length** or free length, which we denote by l_0. If its length is made longer or shorter than l_0 then the spring will always try to return to its natural length.

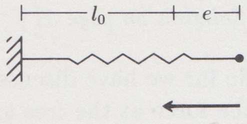

Figure 2
A spring in tension, with extension e. The spring force is directed towards the centre of the spring.

Suppose now that the spring is *extended* (Figure 2), so that its length is greater than its natural length; such a spring is said to be in extension, or alternatively *in tension.* You probably know from experience that if you hold the free end of an extended spring then you will be pulled towards the fixed end. The magnitude of this pulling force is called the **tension** in the spring. The more the spring is extended, the more it pulls; in other words, the tension of the spring increases with its extension.

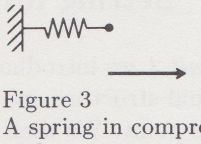

Figure 3
A spring in compression, with compression c. The spring force is directed away from the centre of the spring.

Now suppose that the spring is *compressed* (Figure 3), so that its length is less than its natural length; such a spring is said to be *in compression.* In this case, if you hold the free end of the spring then the spring will push you away from its fixed end. The magnitude of this pushing force is called the **thrust** in the spring. Again, the more the spring is compressed, the more it pushes; the thrust of the spring increases with its compression.

A **perfect spring** is a special model of a real spring. This model is based on the assumption that *the magnitude of the force* (tension or thrust) *exerted by the spring is proportional to the amount of its deformation* (extension or compression). The constant

of proportionality k which relates the force magnitude to the deformation for a perfect spring is known as the **stiffness** of the spring.

The SI units for stiffness are newtons per metre ($\mathrm{N\,m^{-1}}$).

> **The force law for a perfect spring**
>
> A perfect spring is characterized by two positive constants, its *natural length* l_0 and its *stiffness* k. It has zero mass.
>
> Such a spring exerts the following force on any object attached to either of its ends.
>
> (i) When the length of the spring equals l_0, the force is zero.
>
> (ii) When the spring is extended by an amount e (where $e > 0$), so that its length is $l_0 + e$, the force is directed towards the centre of the spring and has magnitude ke (stiffness $\times$ extension).
>
> (iii) When the spring is compressed by an amount c (where $c > 0$), so that its length is $l_0 - c$, the force is directed away from the centre of the spring and has magnitude kc (stiffness $\times$ compression).

In diagrams we label perfect springs with the stiffness only, since it is this which determines the force magnitude exerted by the spring for a given deformation.

The force law for a perfect spring is often known as **Hooke's law**, in honour of Newton's rival, Robert Hooke. In 1678 Hooke declared, in *De Potentia Restitutiva*,

> It is very evident that the Rule or Law of Nature in every springing body is, that the force or power thereof to restore itself to its natural position is always proportionate to the Distance or space it is removed therefrom.

Actually, this rather overstates the case. The 'perfect spring' is only a model: real springs exert forces which have a non-linear dependence on extension or compression, and which also depend on other factors such as the velocities of the ends or the mass of the spring. Nevertheless, many real springs conform fairly closely to the perfect spring model for certain ranges of extension and compression. We shall therefore use a perfect spring as a model for all the springs discussed in this unit, especially as it is a model leading to a differential equation which can be integrated. But you should remember that this is only an *approximation* when comparing any predictions obtained with the real world.

Exercise 1

A perfect spring has a natural length of 0.3 m and a stiffness of $200\,\mathrm{N\,m^{-1}}$. It is attached to a fixed bracket at one end while the other end is attached to a point whose distance d from the fixed bracket is variable (see Figure 4). Determine the magnitude and the direction, relative to the middle of the spring, of the force on the variable end-point when

(i) $d = 0.35\,\mathrm{m}$;

(ii) $d = 0.2\,\mathrm{m}$.

[*Solution on page 34*]

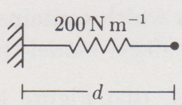

Figure 4

So far we have discussed a spring with one end attached to a fixed point and considered the force at the free end of the spring. However, if you take a spring with an end in each hand and extend it, you will be aware that there are forces exerted by the spring at *both* ends. For a perfect spring in tension, these two forces have equal magnitude and are both directed towards the centre of the spring (see Figure 5). The magnitude of each of these forces is given, as before, by stiffness $\times$ extension. If we denote the length of the extended spring by l then its extension is $l - l_0$, and so the tension T in the spring is equal to $k(l - l_0)$.

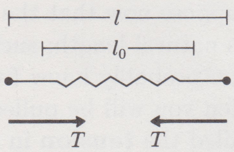

Figure 5

2.2 Setting up the equation of motion

In *Unit 4* we introduced the concept of a *particle* as a material object whose size and internal structure are negligible, so that at any given time the particle is located at a single point. We then analysed the one-dimensional motion of a particle under the

influence of forces such as gravity and air resistance. In the current unit we are again concerned with one-dimensional motion, but now we concentrate on how a particle behaves when connected to one end of a perfect spring.

As before, the analysis is based upon Newton's second law, which states that if a particle of mass m moves along a suitably chosen x-axis and experiences a total force with x-component F, then its acceleration $a = \ddot{x}$ is given by

Unit 4 Subsection 2.3

$$F = ma = m\ddot{x}.$$

Here (and extensively in the remaining mechanics units) we use the dot notation for differentiation with respect to time which was introduced very briefly in Section 1 of *Unit 4*. Thus $\dot{x}$ stands for dx/dt and $\ddot{x}$ represents d^2x/dt^2.

Figure 6 depicts a spring with one end fixed and the other attached to a particle of mass m. The particle is able to move along a horizontal track under the action of the spring force. This system is similar to that of Figure 1(b) in the Introduction, and as mentioned there it might model a railway truck in contact with buffers.

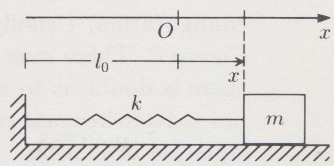

Figure 6

For the sake of simplicity, we assume that the track is frictionless and that the force law for a perfect spring can be used. The spring has stiffness k and natural length l_0. Figure 6 shows the configuration after the spring has been pulled out beyond its free length and then released. We have chosen the x-axis to point from left to right, with the origin at the position occupied by the particle when the spring has its natural length. Clearly the particle's position x varies with time, whereas all the other quantities specified in Figure 6 are constants. The length of the spring at any instant is $l_0 + x$. When $x > 0$ the spring is extended from its natural length and the position x of the particle is equal to the extension e of the spring. When $x < 0$ the spring is compressed by an amount $c = -x$ from its natural length.

The equation of motion is obtained by identifying the total force acting upon the particle and then writing down Newton's second law. As the direction of the spring force depends on whether the spring is extended or compressed, we shall consider these two cases separately. Note that no force acts on the particle when the spring is at its natural length, that is, when $x = 0$.

When the spring is *extended* by an amount $e = x$ $(x > 0)$, it exerts a force on the particle of magnitude

$$T = \text{ stiffness } \times \text{ extension } = ke = kx$$

towards the centre of the spring, that is, in the direction of decreasing x. This is shown on the force diagram in Figure 7, where the particle is represented by a blob. We have not shown the vertical forces (the weight of the particle and the reaction of the fixed horizontal surface) because they are (and remain) equal and opposite and cancel out, there being no motion in this direction. Nor have we shown the force exerted by the spring on the wall at its fixed end: since our concern is with the motion of the particle we need consider only those forces which act directly on the particle. The direction of increasing x is indicated on the force diagram.

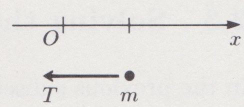

Figure 7

It is now quite simple to derive the equation of motion for the case when the spring is extended. The spring force of magnitude $T = kx$ acts to the left, in the direction of decreasing x, and so the x-component of this force is

$$F = -T = -kx.$$

Hence in this case Newton's second law $ma = F$ becomes

$$m\ddot{x} = -kx,$$

or

$$m\ddot{x} + kx = 0 \quad (\text{for } x > 0). \tag{1}$$

We next consider the case where $x < 0$, and derive the equation of motion when the spring is *compressed* by an amount $c = -x$. In this case the spring force has magnitude

$$T = \text{ stiffness } \times \text{ compression } = kc = -kx$$

and is directed away from the centre of the spring, that is, in the direction of increasing x. This is shown in the force diagram of Figure 8. Now the x-component of this force is T, so Newton's second law gives

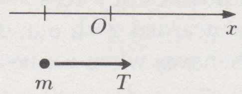

Figure 8

$$m\ddot{x} = T = -kx,$$

or

$$m\ddot{x} + kx = 0 \quad (\text{for } x < 0). \tag{2}$$

Comparing Equations (1) and (2), you can see that *the same equation of motion applies whether the spring is extended or compressed.* This is not accidental—it happens in *all* problems involving motion under the influence of perfect springs, because the spring force is *linear*. In future we shall take advantage of this fact by considering only *one* configuration. Usually, in the rest of the unit, we consider springs to be in tension, as in Figure 6. There may be cases, especially those involving more than one spring, when there is doubt as to whether a given spring is actually in tension or not, but this does not matter because consistent analysis of the 'tension' case will still give the correct equation of motion.

A similar situation arose in *Unit 4* Subsection 4.2, when we were considering motion under linear air resistance.

In the above discussion we measured the position x of the particle from its equilibrium position, where the spring has its natural length. This leads to a particularly simple equation of motion, namely

$$m\ddot{x} + kx = 0. \tag{3}$$

However, there is no reason why we should not measure the position of the particle from any other fixed point, such as the fixed end of the spring. This is equivalent to choosing a different origin for the x-axis, and results in a slightly different equation of motion.

Exercise 2

A spring is attached to a wall at one end of a straight frictionless horizontal track. The other end of the spring is attached to a glider which moves along the track. Model the glider by a particle of mass m and the spring by a perfect spring of natural length l_0 and stiffness k. Find the equation of motion of the glider when the position x of the particle is measured from the fixed end of the spring, as indicated in Figure 9. You may ignore any forces other than that exerted by the spring on the glider.

[Hint: Remember that you need consider only the configuration where the spring is extended. Do not attempt to solve the equation of motion.]

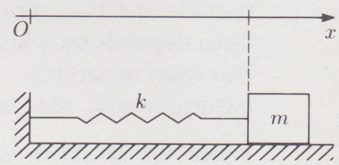

Figure 9

[*Solution on page 34*]

2.3 Solving the differential equation of motion

In the previous subsection we derived Equation (3) as the equation of motion of a particle attached to one end of a horizontal perfect spring whose other end is fixed. *Unit 6* describes a formal method of solving this differential equation, but just looking at the form of the equation gives us a clue about what its solution is.

The only variable quantities to appear in the equation are x and $\ddot{x}$, and the fact that $m\ddot{x} + kx$ is equal to zero (remembering that k and m are positive constants) makes it clear that $x(t)$ and its second derivative must have the same general form but opposite signs, because otherwise the two terms on the left-hand side of the equation could not add up to zero. What functions satisfy this specification? There are at least two which come to mind: the sine and the cosine functions. Hence $x(t) = B\cos\omega t$ and $x(t) = C\sin\omega t$ (where B, C and ω are constants) are contenders. Since we are dealing with a homogeneous linear differential equation, it follows that the sum of these two functions is also a contender.

Exercise 3

Show, by substitution, that $x(t) = B\cos\omega t + C\sin\omega t$ is a solution of the differential equation $m\ddot{x} + kx = 0$, where $\omega^2 = k/m$.

Exercise 4

By using the auxiliary equation, find the general solution of the differential equation

$$m\ddot{x} + kx = 0.$$

This method of solution is described in *Unit 6* Subsection 1.1.

[*Solutions on page 34*]

As you found in Exercise 4, the general solution of Equation (3) is

$$x(t) = B\cos\omega t + C\sin\omega t, \tag{4}$$

where B and C are arbitrary constants which depend upon the initial conditions. The quantity $\omega = \sqrt{k/m}$ is called the (natural) **angular frequency** of the model. Due to the periodic nature of the sine and cosine functions, we have

As the angle ωt is measured in radians, the SI units of the angular frequency ω are radians per second, abbreviated as $\mathrm{rad\,s^{-1}}$.

$$\begin{aligned} x(t) &= B\cos\omega t + C\sin\omega t \\ &= B\cos(\omega t + 2n\pi) + C\sin(\omega t + 2n\pi), \end{aligned}$$

where n is any integer. In other words,

$$\begin{aligned} x(t) &= x\left(t + \frac{2n\pi}{\omega}\right) \\ &= x(t + n\tau), \quad \text{where } \tau = \frac{2\pi}{\omega}. \end{aligned}$$

Thus the motion repeats itself every $\tau = 2\pi/\omega$ seconds. This time, called the **period**, corresponds to one complete cycle of the motion. A graphical representation of Equation (4), over about 2 cycles, is shown in Figure 10. Note that $\dot{x}$ is represented by the slope of the graph at any point, so that $\dot{x} = 0$ when $x = \pm A$, and $|\dot{x}|$ is greatest when $x = 0$.

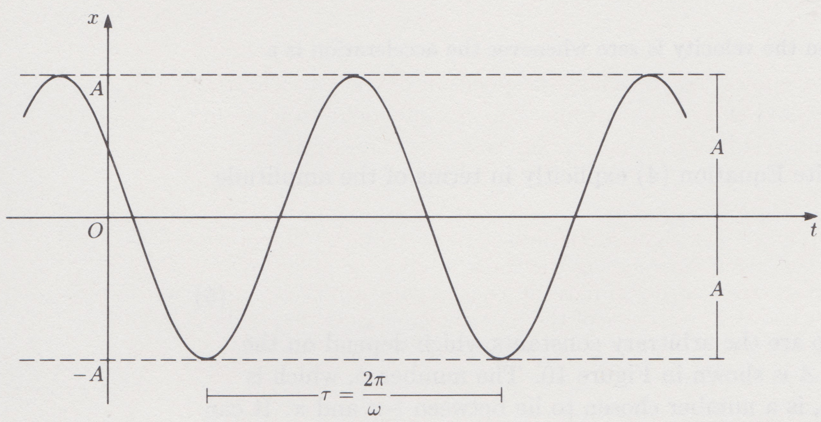

Figure 10

Clearly, the motion of the particle is oscillatory (alternatively positive and negative) and symmetrical about the mean (equilibrium) position. Motion of this type, in which $x(t) = B\cos\omega t + C\sin\omega t$, is known as **simple harmonic motion**. The quantity A indicated in Figure 10 is the maximum departure from the mean position and is called the **amplitude** of the motion. Since every feature of the motion repeats at time intervals of duration τ, the amplitude A is constant. Since τ is the time for one cycle, the number of cycles per unit time, f, is equal to $1/\tau$; this is called the (natural) **frequency** of the model. Note that

The terms 'period' and 'amplitude' were also introduced in the context of the experiment in Section 1, though there the amplitude decreased with time.

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi},$$

so that the angular frequency ω (in $\mathrm{rad\,s^{-1}}$) is equal to 2π times the frequency f (in Hz).

The SI unit for frequency is the *hertz*, which equals 1 cycle per second and is abbreviated as Hz. $1\,\mathrm{Hz} = 1\,\mathrm{s^{-1}}$.

We shall now demonstrate how particular values may be found for the arbitrary constants B and C when specific initial conditions are given. The approach required is that of *Unit 6* Section 3.

Example 1

In a model of a vibrating system, the position of a particle is given by

$$x = B\cos 10t + C\sin 10t.$$

Deduce the values of the constants B and C, given that $x(0) = 0.2$ and $\dot{x}(0) = 1$.

Solution

Since $x(t) = B\cos 10t + C\sin 10t$, we have

$$\dot{x}(t) = -10B\sin 10t + 10C\cos 10t.$$

Substituting $t = 0$ into these equations and using the given initial conditions gives

$$x(0) = B = 0.2,$$
$$\dot{x}(0) = 10C = 1.$$

Hence $B = 0.2$ and $C = 0.1$, giving the particular position function

$$x(t) = 0.2\cos 10t + 0.1\sin 10t. \quad \square$$

Exercise 5

For a particular case of the model shown in Figure 6 (page 9), the spring stiffness is $k = 200$ and the particle mass is $m = 0.5$. The system is set in motion when $t = 0$, at which time $x = 0.3$ and $\dot{x} = 2$. Find

(i) the angular frequency ω and the period τ;

(ii) the frequency f;

(iii) the values of the arbitrary constants B and C in Equation (4).

Exercise 6

Show that in simple harmonic motion the velocity is zero whenever the acceleration is a maximum or minimum.

[*Solutions on page 34*]

There is an alternative way to write Equation (4) explicitly in terms of the amplitude A, namely,

$$\begin{aligned} x(t) &= B\cos\omega t + C\sin\omega t \\ &= A\cos(\omega t + \phi), \end{aligned} \tag{5}$$

where A is positive. Here A and ϕ are the arbitrary constants which depend on the initial conditions. The amplitude A is shown in Figure 10. The number ϕ, which is called the **phase angle** or **phase**, is a number chosen to lie between $-\pi$ and π. It can be shown that

$$A = \sqrt{B^2 + C^2} \tag{6}$$

and

$$\phi = \begin{cases} \arccos(B/A) & C \le 0, \\ -\arccos(B/A) & C > 0. \end{cases} \tag{7}$$

In *Unit 5* we derived these relations by using phasors. Alternatively, they may be obtained as follows:

Unit 5 Subsection 4.3, Example 4

$$\begin{aligned} x &= A\cos(\omega t + \phi) \\ &= A(\cos\omega t\cos\phi - \sin\omega t\sin\phi) \\ &= (A\cos\phi)\cos\omega t + (-A\sin\phi)\sin\omega t. \end{aligned}$$

In order that this should be equal to $B\cos\omega t + C\sin\omega t$ we must have, comparing the coefficients of $\cos\omega t$ and $\sin\omega t$,

$$B = A\cos\phi \quad \text{and} \quad C = -A\sin\phi.$$

Hence

$$B^2 + C^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2,$$

and so $A = \sqrt{B^2 + C^2}$, taking the positive square root because A is positive. We can now find the phase angle ϕ by noting that the equations

$$A\cos\phi = B \quad \text{and} \quad A\sin\phi = -C \tag{8}$$

describe the transformation from Cartesian coordinates $(B, -C)$ to polar coordinates $[A, \phi]$ (see Figure 11). From this figure we see that Equation (7) holds.

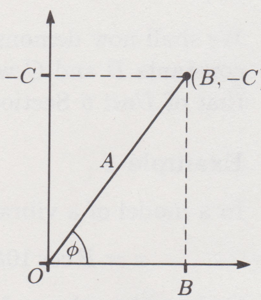

Figure 11

Exercise 7

In Exercise 5 you showed that the particular oscillation described there is represented by the equation

$$x = 0.3\cos 20t + 0.1\sin 20t.$$

Find the amplitude and phase of this oscillation.

[*Solution on page 34*]

Figure 12 is a copy of part of the curve in Figure 10 near to $t = 0$. Its main purpose is to show that increasing the value of ϕ would have the effect of shifting the $x(t)$ curve bodily leftwards parallel to the t-axis by an amount equal to the change in ϕ divided by ω.

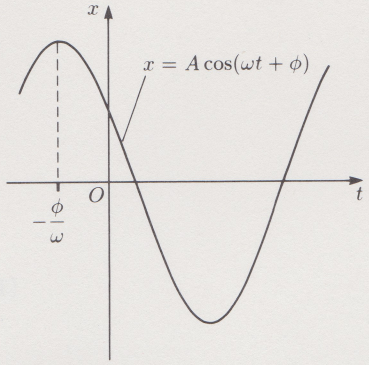

Figure 12

The following box summarizes the main features of simple harmonic motion.

Simple harmonic oscillations

The general solution of the differential equation

$$\ddot{x} + \omega^2 x = 0$$

can be written in the form

$$x(t) = B\cos\omega t + C\sin\omega t,$$

or alternatively as

$$x(t) = A\cos(\omega t + \phi),$$

where

$$A = \sqrt{B^2 + C^2}$$

and

$$\phi = \begin{cases} \arccos(B/A) & C \le 0, \\ -\arccos(B/A) & C > 0. \end{cases}$$

(i) ω is the *angular frequency.*

(ii) The *period* (time for one complete cycle) of the oscillations is

$$\tau = \frac{2\pi}{\omega}.$$

(iii) The *frequency* (number of cycles per second) is

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi}.$$

(iv) A is the *amplitude* of the oscillations.

(v) ϕ is the *phase angle* of the oscillations.

So far in this unit the equilibrium position of the particle has usually been taken as the origin of the x-axis, as this choice leads to the simplest differential equation of motion. We shall continue to adopt the same approach frequently, here and in later mechanics units. However, in certain situations it is more convenient to choose some other origin, so we shall look briefly at how this affects the mathematical solution obtained.

In Exercise 2 you found the equation of motion for a particle at the end of a horizontal perfect spring when the particle's position is measured from the fixed end of the spring (see Figure 13). The equation of motion was

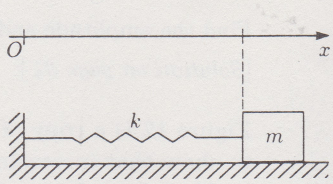

Figure 13

Unit 6 Section 2

$$m\ddot{x} + kx = kl_0, \tag{9}$$

where k, l_0 are respectively the stiffness and natural length of the spring. This differential equation is inhomogeneous, and so its solution is the sum of two parts, a particular solution and the complementary function. A particular solution is given by $kx_p = kl_0$, or

$$x_p = l_0,$$

and the complementary function is, as before,

$$x_c = A\cos(\omega t + \phi), \qquad \text{where } \omega = \sqrt{k/m}.$$

The general solution is therefore

$$\begin{aligned} x &= x_p + x_c \\ &= l_0 + A\cos(\omega t + \phi), \end{aligned} \tag{10}$$

whose graph is shown in Figure 14.

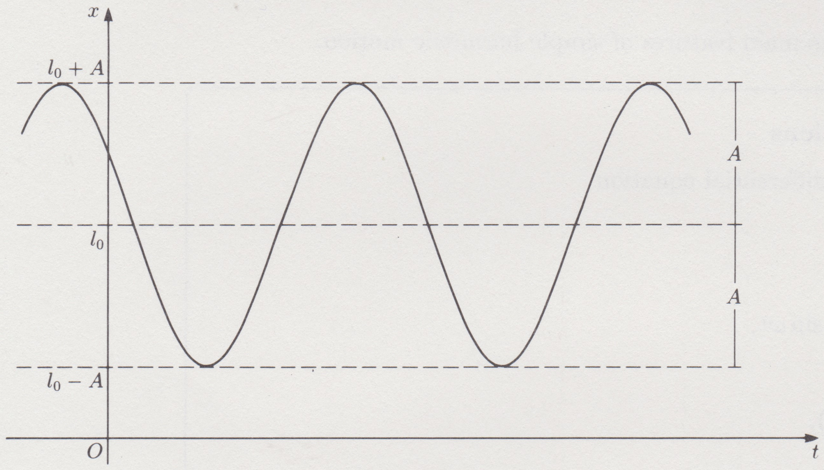

Figure 14

It is clear from this figure that l_0 is the average position of the particle about which the oscillations take place. Apart from this, the motion is identical to that seen earlier, as you might expect. Equation (10) can be written alternatively as

$$x = l_0 + B\cos\omega t + C\sin\omega t, \tag{11}$$

where B and C are given in terms of A and ϕ by Equations (8).

Exercise 8

For the system shown in Figure 13, the particle motion may be described by Equation (11). Find the particular solution which satisfies the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$.

[*Solution on page 34*]

2.4 Oscillations of a vertical spring

In this subsection we shall attempt to model the vibrating system of the experiment described in Section 1, and compare the predictions of the model with the experimental observations. This is a case where the 'perfect spring' is used to model a resilient element (the elastic string of the experimental system) which is not 'perfect'. As a

start, let us take the direction of motion to be vertical (as in the real system). Figure 15 shows the relevant arrangement.

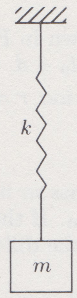

Figure 15

An important difference between the situations described by this figure and by Figure 6 is that here the equilibrium position does *not* correspond to the natural length of the spring as it did in the earlier case. Clearly, if the system in Figure 15 is to be in equilibrium then the spring force must balance the gravitational force of magnitude mg acting on the particle. However, you will see that if we again measure the displacement of the particle from its equilibrium position then the equation of motion takes a familiar form.

Exercise 9

Find the length l_1 of the spring in Figure 15 when the system is in equilibrium.

Exercise 10

In the previous exercise you saw that in equilibrium the spring is extended by an amount mg/k. A vertical spring of stiffness $250\,\mathrm{N\,m^{-1}}$ is to support a metal block of mass 3 kg. Taking $g = 9.81\,\mathrm{m\,s^{-2}}$, find the extension of the spring from its natural length when the block is in equilibrium. If the natural length of the spring is 0.5 m, what will be the mass of the heaviest block it can support if the total length of the spring is limited to 0.8 m?

[*Solutions on page 35*]

In Exercise 9 you showed that the length of the vertical spring in Figure 15 is $l_0 + mg/k$ when the system is in equilibrium. Figure 16(a) shows this same system after the particle has been pulled below its equilibrium position and released. The x-axis has been chosen to point downwards with its origin at the equilibrium position of the particle.

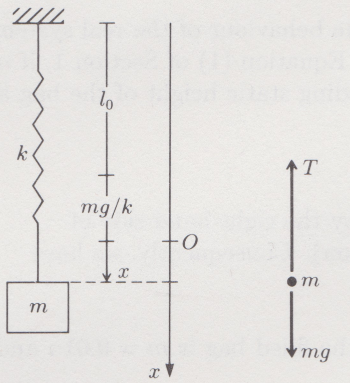

Figure 16

When the particle is at position x, the length of the spring is $x + l_0 + mg/k$. In consequence, the extension is $x + mg/k$ and the magnitude of the spring force is

$$T = k\left(x + \frac{mg}{k}\right) = kx + mg.$$

As can be seen from Figure 16(b), the x-component of the total force acting on the particle is $mg - T$, so Newton's second law takes the form

$$\begin{aligned} m\ddot{x} &= mg - T \\ &= mg - (kx + mg) = -kx, \end{aligned}$$

or $m\ddot{x} + kx = 0$. This is precisely the same equation of motion as that derived earlier for a particle attached to a horizontal spring (Equation (3)). So the motion *about* the equilibrium position here is the same as it was in the horizontal case, but the equilibrium position itself is different. Hence, measured from its equilibrium position the displacement of the particle is $x(t) = A\cos(\omega t + \phi)$, and measured from the fixed end of the spring it is

$$x_1(t) = l_0 + mg/k + A\cos(\omega t + \phi), \tag{12}$$

where $\omega^2 = k/m$ as before.

Exercise 11

The system shown in Figure 16 has a spring of natural length l_0, and when at rest the length of the spring is $l_0 + d$, where $d = mg/k$. Show that the simple harmonic oscillations of this system have period $\tau = 2\pi\sqrt{d/g}$.

Exercise 12

A particle of mass m is suspended from the ceiling by a perfect spring of stiffness k and natural length l_0. If the particle is allowed to oscillate vertically, show that the length x of the spring satisfies the differential equation

$$\ddot{x} + \omega^2 x = \omega^2\left(l_0 + \frac{mg}{k}\right),$$

where $\omega^2 = k/m$. (This is the equation of motion for the particle if the x-axis is chosen to point vertically downwards with its origin at the top of the spring.)

If the system is initially released from rest when the length of the spring is equal to its natural length, show that the subsequent particle motion is given by

$$x(t) = l_0 + \frac{mg}{k}(1 - \cos\omega t).$$

[*Solutions on page 35*]

We now return to the experimental observations from Section 1, where a bag of coins oscillated on the end of an elastic string. It will be assumed that the elastic string can be modelled by an extended perfect spring provided that, as was ensured during the experiment, the string never becomes slack. Then Equation (12) can be applied to the experimental situation.

If the fixed end of the string is at a distance H above the floor then, as Figure 17 shows, the displacement of the bag above the floor is

$$x_2(t) = H - x_1(t) = H - l_0 - mg/k - A\cos(\omega t + \phi). \tag{13}$$

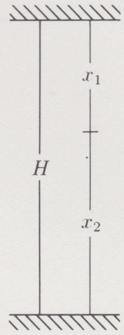

Figure 17

This equation is a prediction of the model about the behaviour of the real system. Let us see how it compares with reality. As you saw in Equation (1) of Section 1, if n is the number of coins in the bag and h_n is the corresponding static height of the bag above the floor, then we have the approximate equation

$$h_n = 1.78 - 0.0430n, \qquad \text{for } 0.7 < h_n < 1.5.$$

According to the model, the static height is given by the right-hand side of Equation (13) with $A = 0$ (since there is no vibration). Consequently, we have

$$h_n = 1.78 - 0.0430n = H - l_0 - mg/k.$$

Since each coin has a mass of 0.01 kg, the mass of the filled bag is $m = 0.01n$ and

$$h_n = 1.78 - 0.0430n = H - l_0 - 0.01ng/k.$$

Equating coefficients of the variable n gives $0.0430 = 0.01g/k$ or, taking $g = 9.81\,\mathrm{m\,s^{-2}}$,

$$k = \frac{0.01 \times 9.81}{0.0430} \simeq 2.28\ \mathrm{N\,m^{-1}}.$$

The simple harmonic model predicts an angular frequency $\omega = \sqrt{k/m}$, where $m = 0.01n$, and the corresponding period is

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 0.2\pi\sqrt{\frac{n}{k}}.$$

Calculating this for $n = 10, 15, 20$ (to make sure of being within the 'linear' range of the string extension even during vibrations), we obtain the approximate figures shown in Table 1. Here τ is the period predicted by the model and τ_{expt} is the corresponding experimental value, as given in Table 2 of Section 1.

The predicted figures are slightly on the high side when compared with the observed values, the greatest error being less than 6%.

Table 1

n	10	15	20
τ	1.32	1.61	1.86
τ_{expt}	1.25	1.6	1.8

The model also predicts that τ^2 is proportional to the number of coins n, and you showed in Exercise 1 of Section 1 that this is approximately the case for the experimental data.

To sum up, the predictions of the perfect spring model are:

(i) a period which remains constant throughout the motion with a particular number of coins in the bag;

(ii) a period which increases as the square root of number of coins in the bag;

(iii) a constant amplitude motion about the equilibrium position once the system is released with particular initial conditions.

Of these three, the first two agree quite well with experimental observations. The third does not; all one can say is that the prediction holds only for very short periods of time and not accurately even then (see Figure 3 in Section 1). The perfect spring model is therefore inadequate in this respect as a representation of the elastic string. One possibility is that the discrepancy is due partly to the model's neglect of frictional effects such as air resistance. This is a matter which needs more attention, and it will be investigated in the next unit.

2.5 Systems with two springs

This subsection extends the treatment of simple oscillatory systems by applying the modelling techniques used so far to systems whose model has a pair of perfect springs.

Exercise 13

The model system shown in Figure 18 features a particle attached to two perfect springs and in contact with a frictionless horizontal track. The distance between the two fixed vertical surfaces is 0.8 m. Spring 1 has a natural length of 0.3 m and a stiffness of $200\,\mathrm{N\,m^{-1}}$. Spring 2 has a natural length of 0.4 m and a stiffness of $300\,\mathrm{N\,m^{-1}}$. The particle has a mass of 2 kg (and, being a particle, its linear dimensions may be neglected). Determine the position, relative to the fixed surfaces, at which the particle can remain at rest.

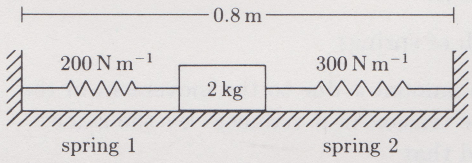

Figure 18

Exercise 14

Consider the linear oscillations of the dynamical system described in Exercise 13. If the x-axis is chosen pointing from left to right with its origin at the particle's equilibrium position, show that the equation of motion of the particle is

$$\ddot{x} + 250x = 0.$$

[*Solutions on page 35*]

As illustrated by the previous exercise, for two-spring systems the simplest equation of motion again arises when the particle's displacement is measured from its equilibrium position. However, in order to find this equation of motion we do not need to locate the equilibrium position explicitly.

Example 2

Figure 19 (overleaf) shows a model system consisting of a particle attached to two springs of different stiffnesses. Friction may be neglected. Derive the equation of motion for this system if the x-axis is chosen pointing to the right with its origin at the equilibrium position of the particle.

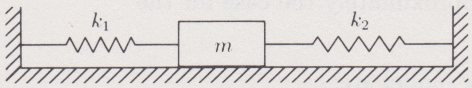

Figure 19

Solution

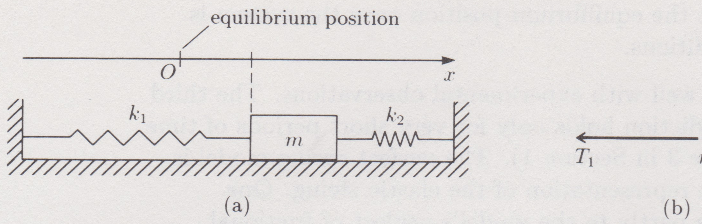

Figure 20

Figure 20(a) shows the particle displaced to the right from its equilibrium position, and Figure 20(b) shows the directions of the spring forces, assuming that both are in tension. We use the suffix 1 for the left-hand spring and the suffix 2 for the right-hand spring. In the equilibrium position, the magnitudes of both spring forces will be the same; we denote this magnitude by T_0. In moving the particle to the right, the magnitude of the tension in spring 1 will be increased (because the spring gets longer). This is a similar case to that of the single vertical spring shown in Figure 15 (page 15). There the equilibrium spring tension T_0 was equal to mg; here the value of T_0 is unspecified, but we can say that in the equilibrium position the extension of spring 1 is T_0/k_1. When the spring is extended by a further distance x (as in Figure 20(a)) then its total extension is $x + T_0/k_1$, and so

$$T_1 = k_1(x + T_0/k_1) = T_0 + k_1 x.$$

Notice that as the particle is displaced by a distance x to the right, the tension in the left-hand spring increases by an amount $k_1 x$, that is,

(stiffness of spring) $\times$ (increase in length of spring).

At the same time, spring 2 is made shorter by an amount x due to the movement of the particle to the right, so that the magnitude of the tension in it reduces. By the same sort of argument as was used for spring 1, we find that

$$T_2 = T_0 - k_2 x.$$

We can now write down the equation of motion for the particle. From Figure 20, the x-component of the total force acting on the particle is $T_2 - T_1$, so Newton's second law gives

$$\begin{aligned} m\ddot{x} &= T_2 - T_1 \\ &= (T_0 - k_2 x) - (T_0 + k_1 x), \end{aligned}$$

or

$$m\ddot{x} + (k_1 + k_2)x = 0. \tag{14}$$

This is the required equation of motion. You will notice that it has the same general (simple harmonic) form as the equation derived for a single spring; here the two springs are acting together, and if one of them were to be removed then we should be back to the single-spring equation. Note that the motion predicted for the particle in this two-spring system is the same as that due to a separate single spring with stiffness $k_1 + k_2$. □

We conclude this section by asking you to investigate two-spring systems which are aligned vertically rather than horizontally.

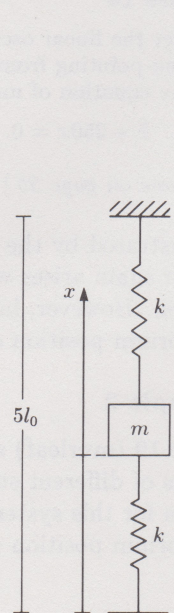

Figure 21

Exercise 15

A particle of mass m oscillates vertically under the influence of gravity and of two identical springs, each of stiffness k and natural length l_0 (see Figure 21). The springs are aligned

vertically, with one attached to the floor and the other attached to the ceiling of a room of height $5l_0$.

(i) Show that the height x of the particle at time t during the vertical oscillations satisfies the differential equation

$$m\ddot{x} + 2kx = 5kl_0 - mg.$$

(ii) Hence find the height at which the particle could remain in equilibrium.

(iii) Find the period τ of the particle's oscillations.

Suppose that the particle is initially (at $t = 0$) released from rest at height $3l_0$.

(iv) Find an expression for the height x of the particle at time t.

(v) What are the minimum and maximum heights of the particle during its oscillations?

Exercise 16

For the model system shown in Figure 22, write down the equation satisfied by the forces which act on the particle when it is in its equilibrium position, assuming that both springs are in tension. Write down the changes to the tensions in the springs when the particle is moved a distance x upwards from its equilibrium position. Hence develop the equation of motion for the vertical oscillations of the particle, where the displacement x is measured upwards from the equilibrium position of the particle. (The argument used in the solution to Example 2 will be useful here.)

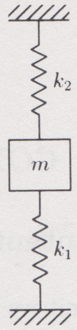

Figure 22

[*Solutions on page 36*]

Summary of Section 2

In this section a perfect spring has been used to model vibrating systems.

1. A **perfect spring** is characterized by two positive constants: its **natural length** and its **stiffness**. It obeys **Hooke's law**, exerting the following force on any object attached to either of its ends.

   (i) When the spring is extended the force is directed towards the centre of the spring and has magnitude (**tension**) equal to stiffness × extension.

   (ii) When the spring is compressed the force is directed away from the centre of the spring and has magnitude (**thrust**) equal to stiffness × compression.

2. If a particle in one-dimensional motion is attached to perfect springs then the equation of motion of the particle can be derived by considering just one configuration in which each spring is designated as being either extended or compressed.

3. The equation of **simple harmonic motion** is

   $$\ddot{x} + \omega^2 x = 0,$$

   whose general solution can be written either as

   $$x = B\cos\omega t + C\sin\omega t$$

   or as

   $$x = A\cos(\omega t + \phi).$$

   The constant ω is called the **angular frequency** and the constant ϕ the **phase angle** of the oscillations. The **period** of the oscillations (time for one complete cycle) is $\tau = 2\pi/\omega$. The **frequency** (number of cycles per second) is $f = 1/\tau = \omega/2\pi$. The **amplitude** of the oscillations is A. The constants A, ϕ and B, C are related via the equations

   $$A = \sqrt{B^2 + C^2}$$

   and

   $$\phi = \begin{cases} \arccos(B/A) & C \leq 0, \\ -\arccos(B/A) & C > 0. \end{cases}$$

4. For a particle in simple harmonic motion, the equation $\ddot{x} + \omega^2 x = 0$ describes the motion when the origin of the x-axis is chosen to be at the equilibrium position of the particle. If another origin is chosen then the equation of motion becomes

$$\ddot{x} + \omega^2 x = \omega^2 x_e,$$

where $x = x_e$ is the particle's equilibrium position. For a particle of mass m attached to one end of a perfect spring of stiffness k whose other end is fixed, the angular frequency ω is given by $\omega^2 = k/m$ for either vertical or horizontal motion. Here gravity is taken into account in the vertical case, but other forces are ignored.

3 The conservation of energy

3.1 The potential energy function

So far in this unit we have concentrated on the motion of a particle which is attached to a perfect spring or springs. We have analysed this motion by using Newton's second law directly. However, there is another technique which is sometimes more convenient for such systems, as well as for other systems whose forces *depend only on the position* of the particle. Such forces are very common. For example, the force acting on a steel pin due to the attraction of a magnet depends upon the distance between the pin and the magnet. In fact, the magnitude of this force increases as the pin is brought closer to the magnet. On the other hand, a dog tethered to a tree by a piece of elastic feels a force which, once the elastic is taut, increases as the dog moves away from the tree. In either case, considered as a one-dimensional situation, the x-component F of the force depends on position, that is,

$$F = F(x).$$

We shall assume that the form of the function $F(x)$ is known, and use this knowledge to answer questions about the motion. Note that it is possible here for $F(x)$ to be a constant function, so in saying that 'F depends only on position' we really mean that F depends *at most* on position; it does not depend directly upon other variables of the motion such as the velocity v, the acceleration a or the time t.

It is natural to begin by writing down Newton's second law, $ma = F$. After putting $a = \dot{v} = dv/dt$ and $F = F(x)$, we have

$$m\frac{dv}{dt} = F(x).$$

Since the force is a function of position x, we use the chain rule

$$\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$$

to rewrite Newton's second law in the form

$$mv\frac{dv}{dx} = F(x).$$

The formula $a = v\dfrac{dv}{dx}$ was introduced in *Unit 4* Subsection 1.5.

This differential equation can be solved by separation of variables, giving

$$m\int v\,dv = \int F(x)\,dx + E,$$

that is,

$$\tfrac{1}{2}mv^2 = \int F(x)\,dx + E,$$

where E is a constant of integration. In other words, we have

$$\tfrac{1}{2}mv^2 + U(x) = E, \qquad (1)$$

where

$$U(x) = -\int F(x)\,dx. \qquad (2)$$

Equation (1) is called the **law of conservation of mechanical energy.** The term $T = \frac{1}{2}mv^2$ is called the **kinetic energy** of the particle and depends on the velocity of the particle, whereas $U = U(x)$ is known as the **potential energy** and depends on the force acting upon the particle. Equation (1) states that the sum $T + U$ of kinetic and potential energies is constant throughout the particle's motion.

Alternatively, U depends on the position of the particle, since the force is itself position-dependent.

Once the potential energy function $U(x)$ is known, the law of conservation of mechanical energy allows us to express the velocity v of the particle in terms of its position x without having to integrate Newton's second law afresh; in cases where the force is a function of position, this law is completely equivalent to Newton's second law. Applications of the conservation law are considered in Subsection 3.2. For the remainder of this subsection we ask you to concentrate on the process of finding the potential energy function $U(x)$ for a known force with x-component $F(x)$.

The potential energy function is defined in Equation (2) as an indefinite integral, so it is defined only to within a constant of integration. In practice, however, the value of this constant is unimportant, because it can be absorbed into the constant E appearing in Equation (1). The constant of integration in the definition of the potential energy function can therefore be chosen to have any convenient value. (You saw a similar situation in *Unit 2*, where the integrating factor for a first-order linear differential equation was defined in terms of an indefinite integral.) In other words, we can choose the potential energy function $U(x)$ to be zero at any convenient point: this point is called the **datum** of the potential energy function. Thus if x_0 is the chosen datum then $U(x_0) = 0$.

Unit 2 Subsection 4.2

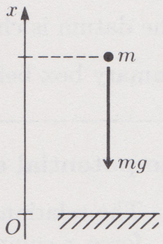

Figure 1

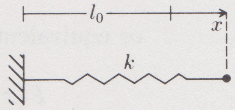

Figure 2

Exercise 1

(i) Using an upward-pointing x-axis, verify that the potential energy function for the force of gravity is

$$U(x) = mgx,$$

when the datum for this function is taken to be the origin $x = 0$ (see Figure 1).

(ii) If the extension of a perfect spring of stiffness k is denoted by x, verify that the potential energy function for the spring force is

$$U(x) = \tfrac{1}{2}kx^2,$$

where the datum is taken to be the point of zero extension, $x = 0$ (see Figure 2).

Exercise 2

Find the potential energy function for each of the following forces (where a, b are constants). State in each case the datum which you are using.

(i) $F(x) = -ax^2$

(ii) $F(x) = b/x^2$

Exercise 3

(i) Suppose that the potential energy functions $U_1(x)$ and $U_2(x)$ correspond to the forces $F_1(x)$ and $F_2(x)$ respectively. Show that the potential energy function corresponding to the force

$$F(x) = F_1(x) + F_2(x)$$

is

$$U(x) = U_1(x) + U_2(x).$$

(ii) Using the result of part (i) and your answers to Exercise 2, write down a potential energy function for the force

$$F(x) = -ax^2 + b/x^2,$$

where a, b are constants.

Exercise 4

(i) Use the definition of the potential energy function $U(x)$ to show that

$$F(x) = -\frac{dU}{dx}.$$

(ii) Find the force which gives rise to the potential energy function

$$U(x) = ax^2 + bx^{-2},$$

where a, b are constants.

[*Solutions on page 37*]

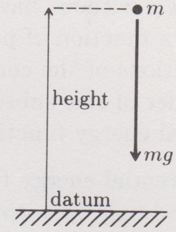

Figure 3

In Exercise 1(i) above you showed that, for a suitable choice of datum ($x = 0$), the gravitational potential energy function is $U(x) = mgx$ if the x-axis is directed upwards. A downward-pointing x-axis leads instead to the function $U(x) = -mgx$, but in either case the potential energy arising from the gravitational force is given by

$$U = mg \times \text{ height above datum} \quad \text{(see Figure 3).}$$

For the perfect spring force, in Exercise 1(ii) you showed that the potential energy function is

$$U = \tfrac{1}{2} \times \text{ stiffness } \times \text{ (extension)}^2,$$

where the datum is chosen to be at the position where the spring has its natural length.

This expression holds also when the spring is compressed, provided that the compression is considered as a 'negative extension'.

The summary box below for the potential energy function concludes this subsection.

The potential energy function

1. The relationship between the *potential energy function* $U(x)$ and the force x-component $F(x)$ is

$$U(x) = -\int F(x)\,dx,$$

or equivalently,

$$F(x) = -\frac{dU}{dx}.$$

2. The point at which the potential energy function is zero is called the *datum*. This can be chosen to be any convenient point.

3. In particular, the gravitational potential energy is given by

$$U = mg \times \text{ height},$$

where the height is measured from some chosen datum, and the potential energy for a perfect spring is

$$U = \tfrac{1}{2} \times \text{ stiffness } \times \text{ (extension)}^2.$$

3.2 Energy conservation and problem-solving

In the previous subsection we concentrated mainly on the relationship between the potential energy function $U(x)$ and the force $F(x)$. Potential energy is important in Newtonian mechanics because it is a term in the law of conservation of mechanical energy, which is Equation (1) of the previous subsection. Indeed, the definition of potential energy arises from the term involving force in that equation. We have summarized this law below. In the rest of the subsection you will see how it can be used to predict the motion of a particle when we know either the total force $F(x)$ acting on the particle or the equivalent potential energy function $U(x)$.

> **The law of conservation of mechanical energy**
>
> Suppose that a particle of mass m moves along a straight line, and that at time t it has position x and velocity v. If the total force on the particle has an x-component $F(x)$ which depends only on the particle's position, then the quantity
>
> $$T + U = E$$
>
> remains constant throughout the particle's motion, where
>
> $$T(v) = \tfrac{1}{2}mv^2$$
>
> is known as the *kinetic energy* and
>
> $$U(x) = -\int F(x)\,dx$$
>
> is known as the *potential energy.* The constant E is known as the *total mechanical energy* of the motion.

The following points regarding energy and the above law of conservation should be noted.

1. The SI unit for energy is the *joule.* Thus a particle of mass 4 kg moving at a speed of $5\,\mathrm{m\,s^{-1}}$ has a kinetic energy of $\frac{1}{2} \times 4 \times 5^2 = 50$ joules.

   The SI unit for energy is the joule (J). $1\,\mathrm{J} = 1\,\mathrm{kg\,m^2\,s^{-2}}$.

2. The conservation of mechanical energy for one-dimensional motion is applicable only for forces $F = F(x)$ which depend on position alone (and not on time, velocity or any other variable).
3. Even when it is applicable, the conservation of mechanical energy says nothing more and nothing less than Newton's second law; it is just an alternative starting point for solving a problem.
4. The value of the total mechanical energy E depends on the datum chosen for the potential energy function.

Example 1

A stone is thrown vertically upwards with an initial speed $15\,\mathrm{m\,s^{-1}}$. Assuming that gravity is the only force acting on the stone, find

(i) the speed of the stone when its height is 5 m;

(ii) the maximum height of the stone.

Solution

This problem could be solved starting from Newton's second law as in *Unit 4*, but it is more direct to use the law of conservation of mechanical energy.

Unit 4 Subsection 3.3

Choose an x-axis pointing upwards with the origin at the point of projection (see Figure 4). The only force acting on the stone is gravity, which being constant can be considered to be a function of x only.

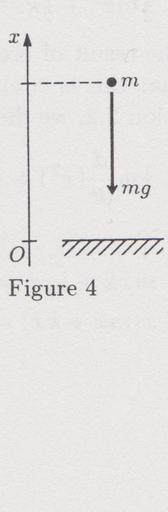

Figure 4

Choosing the origin as the datum, the potential energy is

$$U(x) = mgx \quad \text{(see Exercise 1(i))},$$

so the law of conservation of mechanical energy gives

$$\tfrac{1}{2}mv^2 + mgx = E.$$

Initially $v = 15$ when $x = 0$, and so

$$E = \tfrac{1}{2}m \times 15^2 + m \times 9.81 \times 0 = 112.5m.$$

Hence throughout the motion we have

$$\tfrac{1}{2}v^2 + gx = 112.5. \tag{3}$$

(i) When $x = 5$, Equation (3) gives

$$\tfrac{1}{2}v^2 + 9.81 \times 5 = 112.5.$$

Hence

$$v^2 = 2 \times (112.5 - 9.81 \times 5) = 126.9,$$

so the speed of the stone at height $x = 5$ is

$$|v| = \sqrt{126.9} \simeq 11.3\,\mathrm{m\,s^{-1}}.$$

Notice that this speed is the same whether the stone is moving up or down. The sign of the velocity will, of course, depend on its direction of motion.

(ii) At maximum height, we have $v = dx/dt = 0$. Substituting $v = 0$ into Equation (3) produces

$$gx = 112.5$$

or $$x = 112.5/9.81 \simeq 11.5,$$

so the maximum height attained by the stone is 11.5 metres. □

The above problem is typical of many that can be solved by using the law of conservation of mechanical energy. Notice that it involves a force, gravity, which depends only on position. This is crucial, because the law of conservation of mechanical energy is applicable only if the force is a known function of position. For forces which depend on time or velocity, it is simply not possible to find a potential energy function $U(x)$ such that $\frac{1}{2}mv^2 + U(x) = \text{constant}$.

The problem has a second feature making it especially suited to the methods of this section, namely, it asks for a speed at a given position and a position at a given speed. Problems of this type, which ask us to relate position and velocity rather than to find a position or a velocity at a given time, are ideal for the application of energy conservation, because the conservation law gives an immediate relationship between position x and velocity v.

Exercise 5

(i) A marble, initially at rest, is dropped from the Clifton Suspension Bridge and falls into the River Avon, 77.0 metres below. Assuming that the only force acting on the marble is the force of gravity, use energy conservation to estimate the speed of the marble just before it hits the water. How far has it fallen when its speed reaches $20\,\mathrm{m\,s^{-1}}$? (Choose a downward-pointing x-axis with origin at the point of release, and take this origin as the datum of potential energy. Take care with the sign of the potential energy function.)

You considered this situation previously in Example 1 and Exercise 7 of *Unit 4* Section 3.

(ii) Why can the law of conservation of mechanical energy not be used if air resistance is to be taken into account in the situation above?

[*Solution on page 37*]

Conservation of energy can be applied to particle–spring systems as well as to problems involving gravity. For example, for the system shown in Figure 5 the law of conservation of energy takes the form

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = E, \tag{4}$$

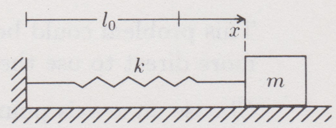

Figure 5

using the result of Exercise 1(ii) for the potential energy of a perfect spring. In order to show that this is equivalent to the simple harmonic equation of motion, derived in Subsection 2.2, we differentiate Equation (4) with respect to t. This gives

$$\tfrac{1}{2}m\frac{d}{dt}(\dot{x}^2) + \tfrac{1}{2}k\frac{d}{dt}(x^2) = 0,$$

leading to

$$m\dot{x}\ddot{x} + kx\dot{x} = 0,$$

or $$\dot{x}(m\ddot{x} + kx) = 0.$$

$$\begin{aligned}\frac{d}{dt}(x^2) &= \frac{d}{dx}(x^2)\frac{dx}{dt}\\ &= 2x\dot{x},\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(\dot{x}^2) &= \frac{d}{dt}(v^2)\\ &= \frac{d}{dv}(v^2)\frac{dv}{dt}\\ &= 2v\dot{v} = 2\dot{x}\ddot{x}.\end{aligned}$$

We can neglect the possibility $\dot{x} = 0$, which implies that the particle is stationary, and hence obtain

$$m\ddot{x} + kx = 0,$$

as before (this is Equation (3) of Section 2).

Exercise 6

A particle of mass $m = \frac{1}{2}$ moves along a horizontal straight frictionless track. The particle is attached to the end of a perfect spring of stiffness $k = 2$, the other end of the spring being fixed (see Figure 6).

Initially the extension of the spring is $x = 2$ and the particle's velocity is $v = -3$. What is the particle's maximum speed during the subsequent motion, and at what positions is it momentarily at rest? (Throughout this question, all quantities are measured in the appropriate SI units.)

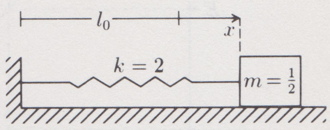

Figure 6

Exercise 7

A particle of mass m is attached to one end B of a perfect spring AB of natural length l_0 and stiffness $k = 10mg/l_0$, where g is the magnitude of the acceleration due to gravity. The spring is hung vertically with the end A fixed (see Figure 7).

The particle is pulled vertically downwards until the spring's length is $\frac{5}{4}l_0$ and then released from rest. By using the principle of conservation of energy, find the speed of the particle when the spring is of length l_0 in the subsequent motion.

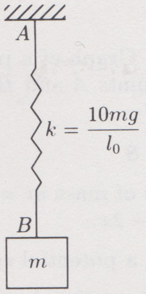

Figure 7

[*Solutions on page 38*]

As you have seen in the above example and exercises, the law of conservation of mechanical energy provides a convenient method for finding the relationship between the velocity v and position x whenever the force acting on the particle is a function of position. However, it can also be used to find the relationship between position and time in these cases by rearranging the law

$$\tfrac{1}{2}mv^2 + U(x) = E$$

and putting $v = dx/dt$ to obtain

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}(E - U(x))}. \tag{5}$$

The sign of the term on the right-hand side of this equation depends on the direction of motion of the particle (but note that the speed $|v|$ of the particle at a given position is independent of the direction of motion). This differential equation can be solved by the separation of variables method to give

$$t = \pm\int\sqrt{\frac{m}{2(E - U(x))}}\,dx + c.$$

From Equation (5) we see that motion is possible only in regions for which

$$E - U(x) \geq 0,$$

and that the particle is momentarily at rest ($v = 0$) at points for which

$$E - U(x) = 0.$$

At such points, called **turning points**, the particle changes its direction of motion. This is illustrated in Figure 8 overleaf. A particle, moving under the action of a force which gives rise to the illustrated potential energy function $U(x)$ and with total energy E, would oscillate backwards and forwards between the two turning points A and B, changing its direction of motion each time that it reached one of these points.

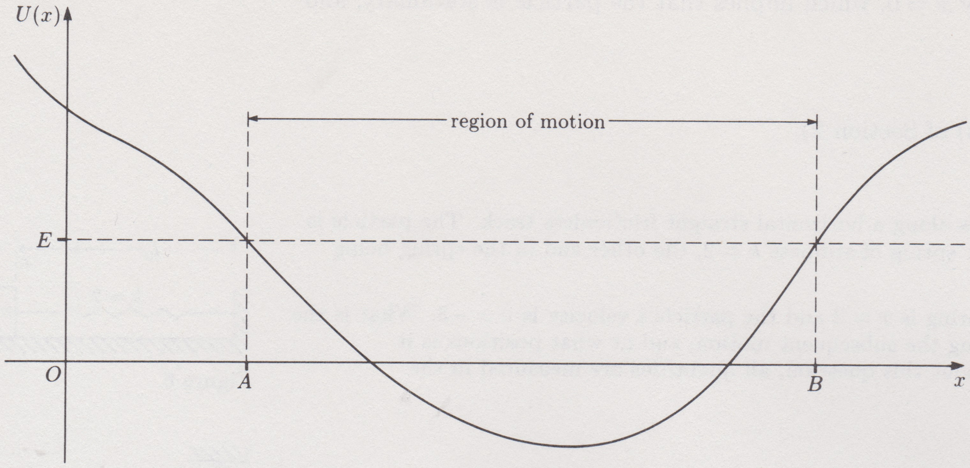

Figure 8 Graph of a potential energy function $U(x)$, showing the region of motion and turning points A and B for motion with total mechanical energy E.

Exercise 8

A particle of mass $m = 2$ moves under the influence of a force with x-component $F(x) = 2 - 2x$.

(i) Find a potential energy function $U(x)$ for the above force.

(ii) If the particle is released from rest at $x = -1$, find the total mechanical energy of the particle in the subsequent motion.

(iii) Use the law of conservation of mechanical energy to find an expression for the velocity v of the particle at position x.

(iv) Find the region of motion of the particle and its speed at the mid-point of this region.

[*Solution on page 38*]

Summary of Section 3

1. The relationship between a force with x-component $F(x)$ and the **potential energy function** $U(x)$ is

$$U(x) = -\int F(x)\,dx,$$

or equivalently,

$$F(x) = -\frac{dU}{dx}.$$

The point at which the potential energy function is zero is called the **datum**. This can be chosen to be any convenient point.

2. In particular, the gravitational potential energy is

$$U = mg \times \text{ height},$$

where the height is measured from the chosen datum.

3. The potential energy of a perfect spring is

$$U = \tfrac{1}{2} \times \text{ stiffness } \times \text{ (extension)}^2.$$

4. The **law of conservation of mechanical energy** states that if the total force acting on a particle depends only upon the particle's position then the quantity

$$\tfrac{1}{2}mv^2 + U(x) = E$$

remains constant throughout the particle's motion. The quantity

$$T = \tfrac{1}{2}mv^2$$

is known as the **kinetic energy** of the particle and the constant E as its **total mechanical energy**.

5. The region of motion of a particle with total mechanical energy E satisfies the inequality

$$E - U(x) \geq 0.$$

The end-points of this region, for which

$$E - U(x) = 0,$$

are called the **turning points** of the motion.

4 The fabulous perfect spring (Television Section)

4.1 Small oscillations about positions of equilibrium

In the first half of this unit we considered the motion of a particle under the action of perfect springs and saw that the motion was a simple harmonic vibration. At the end of the last section you saw that if a particle with total mechanical energy E is acted upon by any force which depends only on position and which gives rise to a potential energy function $U(x)$, and if the particle is in a region between two turning points for which $E - U(x) \geq 0$, then the particle oscillates backwards and forwards between the turning points (see Figure 1).

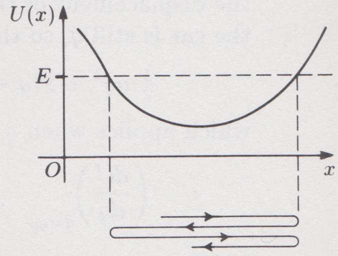

Figure 1

In general this motion is not simple harmonic; in fact, it will be simple harmonic only if that portion of the graph of $U(x)$ which lies between the turning points is part of a parabola. However, you will see in this section that, even for a general potential function, the particle motion can be regarded as approximately simple harmonic for *small* oscillations about positions of equilibrium. In the television programme this is demonstrated by the motion of a toy car on a curved track, as shown in Figure 2. This motion does not take place along a straight line, but the law of conservation of energy may still be applied here.

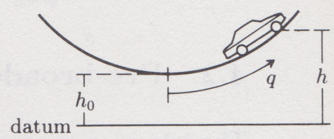

Figure 2

The displacement q of the car from the lowest part of its path is measured along the curve of the track, as might be done with a flexible tape measure. Then the vertical height h above any fixed datum level will be a function of q and so the gravitational potential energy of the car, which is mgh measured from the same datum, is also a function of q, namely, $U(q) = mgh(q)$.

The principle of conservation of energy in these circumstances takes the form

This model ignores the effects of friction and air resistance.

$$\tfrac{1}{2}m\dot{q}^2 + U(q) = E,$$

where $\dot{q}$ is the car's velocity along the track and E is its (constant) total mechanical energy. We have used the symbol q to denote the displacement of the car along the curved track because x is reserved to denote displacement along a straight line. The car will be in equilibrium at the lowest point of the track, which we can choose to be the point at which $q = 0$ without any loss of generality. At this point the potential energy function $U(q)$ has a minimum, so that $U'(0) = 0$ and $U''(0) > 0$. Furthermore, we can choose the datum for the potential energy function so that $U(0) = 0$ (which amounts to choosing $h_0 = 0$ in Figure 2). In the neighbourhood of the equilibrium position, the potential energy function can be approximated by using a Taylor series, giving

The condition $U''(0) > 0$ is not strictly necessary for a minimum (since $U''(0)$ may be zero), but it is satisfied in the majority of cases.

You may have met Taylor series in a prerequisite course. They are discussed in the Preparatory Booklet.

$$\begin{aligned} U(q) &= U(0) + qU'(0) + \frac{1}{2!}q^2U''(0) + \frac{1}{3!}q^3U'''(0) + \cdots \\ &= \frac{1}{2!}q^2U''(0) + \frac{1}{3!}q^3U'''(0) + \cdots \quad \text{(since } U(0) = 0 \text{ and } U'(0) = 0) \\ &\simeq \tfrac{1}{2}q^2U''(0), \end{aligned}$$

provided that q is small. Hence the law of conservation of energy reduces to

$$\tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}q^2U''(0) \simeq E,$$

or $$\tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}kq^2 \simeq E, \qquad (1)$$

where $k = U''(0)$. Apart from the approximation involved, and the use of q rather than x as a coordinate, this is the same equation as that obtained for a particle moving under the action of a perfect spring, namely Equation (4) of Section 3. So in the vicinity of its position of equilibrium the car oscillates simple harmonically with angular frequency

$$\omega \simeq \sqrt{\frac{k}{m}} = \sqrt{\frac{U''(0)}{m}}.$$

Exercise 1

A bead of mass m is threaded onto a frictionless horizontal wire. The bead is attached to a perfect spring of stiffness k and natural length l_0, whose other end is fixed to a point A at height h above the wire (where $h > l_0$). If the position x of the bead is measured from the point on the wire vertically below A, as shown in Figure 3, find the potential energy function $U(x)$. Hence find the period of small oscillations of the bead about its equilibrium position.

[*Solution on page 38*]

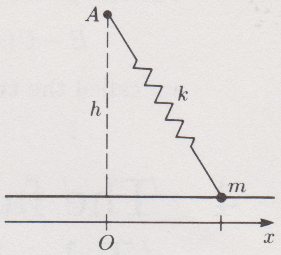

Figure 3

In the television programme the model for small oscillations embodied by Equation (1) is generalized a little, by measuring the displacement q of the car from a position other than the lowest point on the track. If this lowest point is at $q = q_0$ (see Figure 4), then the displacement of the car relative to its equilibrium position is $q - q_0$. The velocity of the car is still $\dot{q}$, so that Equation (1) now becomes

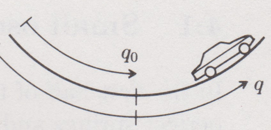

Figure 4

$$\tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}(q - q_0)^2 U''(q_0) \simeq E,$$

which applies when q is close to q_0. The television programme also uses the notation

$$\left(\frac{dU}{dq}\right)_{q=q_0} \quad \text{for} \quad U'(q_0)$$

and

$$\left(\frac{d^2U}{dq^2}\right)_{q=q_0} \quad \text{for} \quad U''(q_0).$$

4.2 Pre-broadcast notes

The television programme illustrates the material in Sections 2 and 3. It begins by demonstrating simple harmonic motion as induced by a spring. The questions posed in the following exercise are discussed in the programme, so you should attempt this exercise before the broadcast.

Exercise 2

Figure 5(a) shows a glider of mass m which slides along a frictionless horizontal air track. The glider is attached to a fixed post by a perfect spring of stiffness k and performs simple harmonic oscillations of period T_1.

We use the symbol T to denote the period here for consistency with the usage in the television programme. Elsewhere in the unit the symbol τ is used for the period (whereas T stands for the tension).

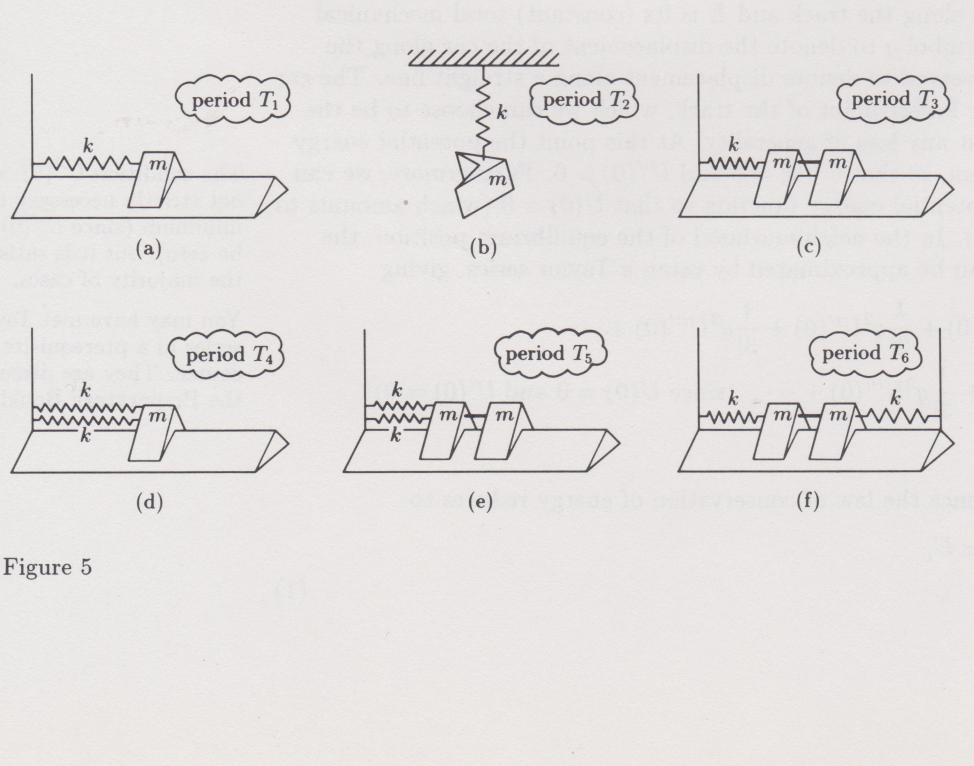

Figure 5

Figures 5(b)–5(f) show different arrangements that can be constructed from gliders of mass m and perfect springs of stiffness k. The symbols $T_2, \ldots, T_6$ denote the periods of oscillation in these systems. Use intuition, or Newton's second law, to state whether

(i) T_1 depends on the amplitude of the oscillations;

(ii) T_2 is less than, equal to, or greater than T_1;

(iii) T_3 is less than, equal to, or greater than T_1;

(iv) T_4 is less than, equal to, or greater than T_1;

(v) T_5 is less than, equal to, or greater than T_1;

(vi) T_6 is less than, equal to, or greater than T_1.

[*The solution to this exercise is given in the television programme.*]

Note that in the television programme the position x of the particle is measured from the fixed post, so that the equation of motion for the system in Figure 5(a) is $m\ddot{x} + kx = kl_0$ as found in Exercise 2 of Section 2. This may be written as $m\ddot{x} = -k(x - l_0)$, which is the form adopted in the programme. Its general solution, $x = l_0 + A\cos(\omega t + \phi)$, was also derived in Section 2.

Section 2, Equation (10)

If possible, *you should watch the television programme now*, before reading on. The following synopsis will then remind you of what you have seen.

TV7

4.3 Programme synopsis

The programme divided naturally into three parts, namely,

Part 1: perfect springs and Newton's second law;
Part 2: perfect springs and energy conservation;
Part 3: energy conservation for a pendulum bob or toy car.

Part 1 of the programme gave a brief review of the properties of perfect springs. Allan Solomon recalled the model discussed in Section 2. The displacement of the particle from the fixed end of the spring was denoted by x (see Figure 6), so that the equation of motion was

References in the television programme to *Unit 4* should be taken as references to Section 3 of this unit.

$$m\ddot{x} = -k(x - l_0),$$

which has the general solution

$$x(t) = A\cos(\omega t + \phi) + l_0.$$

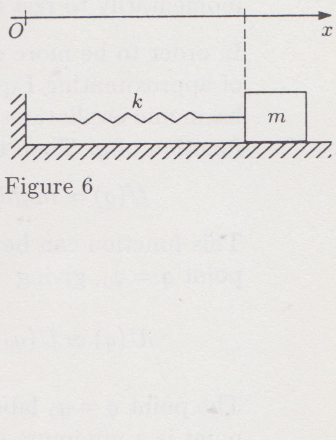

Figure 6

This solution contains four constants: A, ϕ, ω and l_0. But A and ϕ play quite different roles from those of ω and l_0. The amplitude A and phase ϕ depend on how the system was set in motion. On the other hand, the angular frequency ω and the natural length l_0 characterize the system itself and are independent of the initial conditions. This is also true of the period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

The fact that the period does not depend on the amplitude is one of the hallmarks of *simple harmonic* oscillations.

David Broadhurst then showed some real oscillating systems. He concentrated on the systems of gliders and springs which you considered in Exercise 2 of the pre-broadcast notes, and found that

These experimental results should agree with your answers to Exercise 2.

(i) T_1 is independent of amplitude;
(ii) $T_2 = T_1$;
(iii) $T_3 > T_1$;
(iv) $T_4 < T_1$;
(v) $T_5 = T_1$;
(vi) $T_6 = T_1$.

Part 2 of the programme applied the law of conservation of energy,

$$\tfrac{1}{2}m\dot{x}^2 + U(x) = E \quad \text{(constant)},$$

to the system illustrated in Figure 6. The potential energy was found to be

$$\begin{aligned} U(x) &= -\int F(x)\,dx + C \\ &= \int k(x-l_0)\,dx + C \\ &= \tfrac{1}{2}k(x-l_0)^2 + C. \end{aligned}$$

The arbitrary constant C was chosen to be zero, so that the potential energy was zero when the spring had its natural length, l_0. The law of conservation of energy then took the form

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}k(x-l_0)^2 = E \quad \text{(constant)}. \tag{2}$$

It was emphasized that the total energy remains constant while the individual kinetic and potential energies change continually. The potential energy is zero when the spring is at its natural length, and reaches its maximum value at full compression or full extension. On the other hand, the kinetic energy has its maximum value when the spring is at its natural length and is zero at full compression or full extension.

Part 3 of the programme extended the ideas of Part 2 by looking at the motion of a particle along a curved path in a vertical plane, which models the motions of a pendulum bob and of a toy car on a curved track (see Figure 7). The car oscillates to and fro about the lowest point of the track.

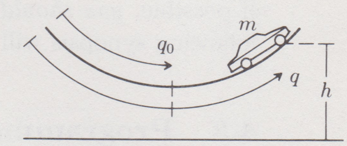

Figure 7

The programme assumes that frictional and air resistance effects can be ignored, and uses the energy conservation equation

$$\tfrac{1}{2}m\dot{q}^2 + mgh = E \quad \text{(constant)}, \tag{3}$$

where m is the mass of the car, h is the height of the car above the floor, q is the distance of the car from one end of the track (measured with a flexible tape measure that follows the curve of the track) and $\dot{q}^2$ is the (speed)^2 of the car.

The oscillations of the car can be interpreted in terms of Equation (3). Suppose that the car is released from rest and starts to roll downhill. Then h decreases, so $\dot{q}^2$ increases and the car picks up speed. This continues until the car begins to climb uphill. Then h increases, so $\dot{q}^2$ decreases and the car loses speed. Eventually it comes momentarily to rest at its original height and the oscillation is ready to begin again.

In order to be more quantitative about this, consideration was given to the possibility of approximating Equation (3) by a form of Equation (2), which is typical of simple harmonic oscillations. In order to do this, the height h of the car was regarded as a function of q. Then the potential energy is also a function of q, that is,

$$U(q) = mgh(q).$$

This function can be approximated by the first three terms of a Taylor series about the point $q = q_0$, giving

$$U(q) \simeq U(q_0) + (q-q_0)\left(\frac{dU}{dq}\right)_{q=q_0} + \tfrac{1}{2}(q-q_0)^2\left(\frac{d^2U}{dq^2}\right)_{q=q_0}$$

Note that this approximate equality is exact if $U(q)$ is a quadratic function (as in the perfect-spring case), because then the Taylor series has no terms of degree higher than two.

The point $q = q_0$ labels the lowest point of the curved track in Figure 7. Since this point is a minimum of $U(q)$, we have

$$\left(\frac{dU}{dq}\right)_{q=q_0} = 0 \quad \text{and} \quad \left(\frac{d^2U}{dq^2}\right)_{q=q_0} \geq 0.$$

In fact, it is assumed that d^2U/dq^2 is strictly positive at $q = q_0$. If it is zero then further terms of the Taylor series must be included.

The datum for the potential energy function was chosen to be $q = q_0$, so that $U(q_0) = 0$. Then Equation (3) for the total mechanical energy becomes

$$\tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}(q-q_0)^2\left(\frac{d^2U}{dq^2}\right)_{q=q_0} \simeq E \tag{4}$$

for q close to q_0. This has the same form as Equation (2), with x replaced by q, l_0 replaced by q_0, and k replaced by $(d^2U/dq^2)_{q=q_0}$. This suggests that the motion of the car on the curved track should be approximately simple harmonic, with period

$$T = \frac{2\pi}{\omega}, \quad \text{where } \omega \simeq \sqrt{\frac{k}{m}} \text{ and } k = \left(\frac{d^2U}{dq^2}\right)_{q=q_0}$$

This approximation was used by Mike Crampin to estimate the angular frequency of small oscillations of a 'simple' pendulum like that shown in Figure 8. It is modelled by a particle at the end of a fixed length l of massless string. The datum for the potential energy function is chosen to be the lowest position of the particle (which is also its equilibrium position). The displacement q is measured from this position too, and so q_0 is zero. Under these circumstances, Equation (4) reduces to Equation (1) of Subsection 4.1.

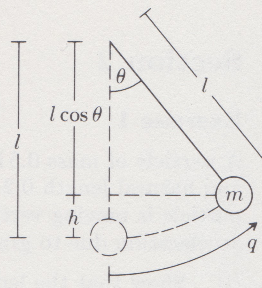

Figure 8

The analysis is restricted to small oscillations, so that the value of q is small and it is reasonable to apply the approximation developed earlier. The gravitational potential energy of the particle is

$$U = mgh = mgl(1-\cos\theta), \qquad \text{where } \theta = q/l.$$

Substituting for θ and writing U as a function of q gives

$$U(q) = mgl\left(1 - \cos\frac{q}{l}\right),$$

whose successive derivatives are

$$\frac{dU}{dq} = mg\sin\frac{q}{l} \quad \text{and} \quad \frac{d^2U}{dq^2} = \frac{mg}{l}\cos\frac{q}{l}.$$

Hence $(d^2U/dq^2)_{q=0} = mg/l$, so that

$$\omega \simeq \sqrt{\frac{mg/l}{m}} = \sqrt{\frac{g}{l}} \quad \text{and} \quad T \simeq 2\pi\sqrt{\frac{l}{g}}.$$

Notice that this result is independent of the mass m.

The following exercises should be attempted after you have viewed the television programme.

Exercise 3

A toy car of mass m moves along a curved track as shown in Figure 7. The height h of the car above its lowest position is given by $h = q^2/(4a)$ for $-2a < q < 2a$, where q is the displacement of the car from its lowest position and a is a positive constant (we have taken $q_0 = 0$ and $h = 0$ at $q = 0$ in Figure 7). The effects of friction and air resistance may be ignored.

The curve which satisfies this equation is called a *cycloid*.

(i) Find the period of small oscillations of the car about its equilibrium position.

(ii) Find the period of large oscillations of the car about its equilibrium position.

Exercise 4

For the simple pendulum in Figure 8, write down an expression for the total mechanical energy E (which is the sum of the kinetic and potential energies) in terms of q. By differentiating this energy equation with respect to time, show that the equation of motion for the pendulum bob is

$$\ddot{q} + g\sin\frac{q}{l} = 0.$$

By using the approximation $\sin\theta \simeq \theta$ for small values of θ, find the period of small oscillations of the pendulum.

[*Solutions on page 38*]

Summary of Section 4

For a system such as a pendulum or a toy car on a curved track, the law of conservation of energy has the form

$$\tfrac{1}{2}m\dot{q}^2 + U(q) = E,$$

where q is the displacement of the pendulum bob along its path or of the car measured along the track. For *small oscillations* about the lowest point of the path $q = q_0$, the potential energy function can be approximated by the first three terms of its Taylor series. This approximation leads to the prediction of simple harmonic motion of angular frequency $\omega \simeq \sqrt{U''(q_0)/m}$ and period $\tau \simeq 2\pi\sqrt{m/U''(q_0)}$.

5 End of unit exercises

Section 2

Exercise 1

A particle of mass 0.5 kg is suspended from the ceiling by a perfect spring of stiffness $50\,\mathrm{N\,m^{-1}}$ and natural length 0.2 m. At time $t = 0$ the length of the spring is its natural length and the particle is moving vertically downwards at a speed of $1\,\mathrm{m\,s^{-1}}$. The magnitude of the acceleration due to gravity is to be taken as $g = 10\,\mathrm{m\,s^{-2}}$.

(i) Show that the length x of the spring satisfies the differential equation

$$\ddot{x} + 100x = 30.$$

(ii) Find the length of the spring in the equilibrium position.

(iii) With the given initial conditions, find the length of the spring in terms of time t.

(iv) Find the period, amplitude and phase of the vibrations of the particle.

(v) When does the particle first reach its lowest point, and what is the maximum length of the spring during the motion?

Exercise 2

A particle of mass m is hung from the ceiling by a perfect spring of stiffness k and natural length l_0. Find the depth of the particle below the ceiling when it is hanging in equilibrium.

The particle is pulled a distance $\frac{1}{4}l_0$ below its equilibrium position and then released from rest. Derive the equation of motion of the particle in terms of its downward displacement x from the equilibrium position. Find the particular solution of this differential equation which satisfies the given initial conditions.

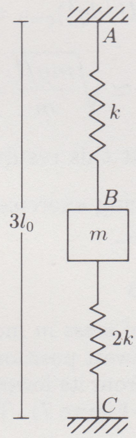

Figure 1

Exercise 3

Figure 1 shows a particle B of mass m attached via two perfect springs to fixed points A and C, where C is a distance $3l_0$ vertically below A. Both springs have natural length l_0. Spring AB has stiffness k, and spring BC has stiffness $2k$.

Find the height of the particle B above C when the system is in equilibrium. Further, obtain the equation of motion for the vertical oscillations of the particle in terms of the height x of the particle above its equilibrium position.

[*Solutions on page 39*]

Section 3

Exercise 4

A particle of mass 3 moves along a straight line and experiences a force which repels it from the origin of position, $x = 0$. This force has x-component $F = 2/x^2$ for $x > 0$. Initially the particle is moving towards the origin, being at the position $x = 10$ with velocity $v = -1$. (All quantities here are measured in the appropriate SI units.)

(i) Find a potential energy function for this force for $x > 0$, stating the datum which you use.

(ii) Find the total mechanical energy of the particle.

(iii) What is the closest point to the origin which is reached by the particle?

(iv) Sketch the graph of the potential energy function, and indicate the region of motion.

Exercise 5

The system in Figure 2 can be used to model a railway truck in contact with buffers. The mass of the truck is $m = 2000\,\mathrm{kg}$ and the stiffness of the buffer springs is $k = 10^5\,\mathrm{N\,m^{-1}}$. If the truck starts from rest at a position where the springs are compressed by an amount 0.1 m, find the speed of the truck when it leaves the buffers. You may neglect friction, and assume that the truck leaves the buffers when the springs have their natural length.

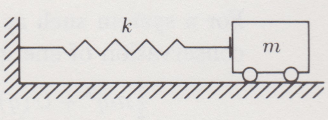

Figure 2

Exercise 6

A bead of mass m moves without friction along a straight wire which is at an angle θ to the vertical (see Figure 3). The bead starts from rest at the point A and slides down to the point B, which is a vertical distance h below A. The displacement of the bead from A, measured along the wire, is denoted by q. The point B is to be taken as the datum for the potential energy function.

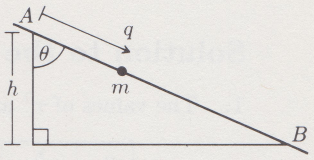

Figure 3

(i) Show that the total mechanical energy of the bead can be written as

$$E = \tfrac{1}{2}m\dot{q}^2 + mg(h - q\cos\theta).$$

(ii) Use conservation of energy to obtain the differential equation

$$\ddot{q} = g\cos\theta \quad (\text{when } \dot{q} \neq 0).$$

(iii) How long does the bead take to slide from A to B?

Exercise 7

A particle of mass 3 moves along a straight line while experiencing a total force $F(x) = 1 - 2x$, where x is the position of the particle on the straight line. Find the potential energy function for this force.

When the particle is at the origin $x = 0$, it has velocity 2. By using the law of conservation of energy, find the region of motion for the particle. (All quantities here are measured in the appropriate SI units.)

[*Solutions on page 40*]

Section 4

Exercise 8

A particle of mass m rests on a frictionless horizontal surface and moves along a straight line perpendicular to a fixed wall. Its distance from the wall is denoted by x. The particle experiences two forces: a force of magnitude A/x^2 directed *towards* the wall and a force of magnitude B/x^{10} directed *away* from the wall (where A and B are positive constants).

(i) What is the equilibrium position of the particle (that is, the value $x = x_0$ for which the particle can remain permanently at rest)?

(ii) What is the potential energy function for the particle?

(iii) If the particle is disturbed slightly from its equilibrium position, what is the period of the subsequent small oscillations? Express your answer in terms of m, A and x_0.

[*Solution on page 41*]

Appendix: Solutions to the exercises

Solution to the exercise in Section 1

1. The values of τ^2 are given in the table below.

n	5	10	15	20	25
τ^2	0.49	1.56	2.56	3.24	4.41

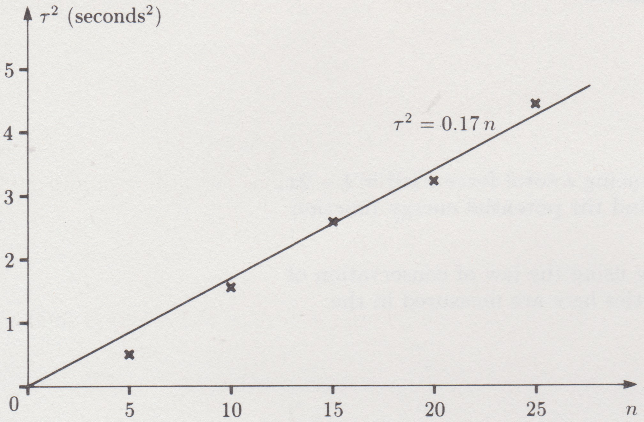

Figure 1

As you can see from Figure 1, a straight line agrees fairly well with the data, the best fit being given by $\tau^2 = 0.17n$.

Solutions to the exercises in Section 2

1. (i) When $d = 0.35$ the spring is extended by an amount $e = 0.05$, so that the magnitude of the spring force is $0.05 \times 200 = 10$ (newtons). Since the spring is in tension, the force on the end-point is directed towards the centre of the spring.

(ii) When $d = 0.2$ the spring is compressed by an amount $c = 0.1$, so that the magnitude of the spring force is $0.1 \times 200 = 20$ (newtons). Since the spring is in compression, the force on the end-point is directed away from the centre of the spring.

2.

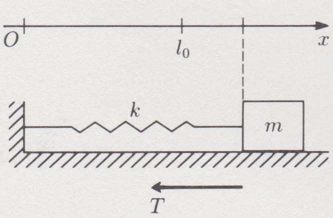

Figure 1

The length of the spring is x. Assuming that the spring is in tension, its extension is $x - l_0$. So the spring force on the particle has magnitude

$$\begin{aligned} T &= \text{stiffness} \times \text{extension} \\ &= k(x - l_0) \end{aligned}$$

and direction towards the centre of the spring, that is, in the direction of decreasing x. So the x-component of this force is $-T$, and Newton's second law gives

$$\begin{aligned} m\ddot{x} &= -T \\ &= -k(x - l_0), \end{aligned}$$

or $m\ddot{x} + kx = kl_0$.

[Notice the similarities, and difference, between this equation of motion and that just derived for displacement from the equilibrium position (Equation (3) of Section 2).]

3. If $x(t) = B\cos\omega t + C\sin\omega t$ then

$$\dot{x}(t) = -B\omega\sin\omega t + C\omega\cos\omega t$$

and $\ddot{x}(t) = -B\omega^2\cos\omega t - C\omega^2\sin\omega t = -\omega^2 x(t)$.

Putting $\omega^2 = k/m$ and writing x for $x(t)$, we have

$$\ddot{x} = -kx/m \quad \text{or} \quad m\ddot{x} + kx = 0.$$

4. We write the differential equation in the form

$$\ddot{x} + \omega^2 x = 0, \quad \text{where } \omega^2 = k/m.$$

Then the auxiliary equation is

$$\lambda^2 + \omega^2 = 0,$$

and hence $\lambda = \pm i\omega$. So the solution is

$$x = e^0(B\cos\omega t + C\sin\omega t),$$

or $x = B\cos\omega t + C\sin\omega t$.

[This is very similar to Example 3 of *Unit 6* Section 1.]

5. (i) The angular frequency is

$$\omega = \sqrt{k/m} = \sqrt{200/0.5} = 20\,\text{rad s}^{-1}.$$

The period is

$$\tau = 2\pi/\omega = 2\pi/20 \simeq 0.314\,\text{s}.$$

(ii) The frequency is

$$f = \omega/(2\pi) = 20/(2\pi) \simeq 3.18\,\text{Hz}.$$

(iii) We have $x = B\cos\omega t + C\sin\omega t$, from which

$$\dot{x} = -B\omega\sin\omega t + C\omega\cos\omega t.$$

Now $x = 0.3$ and $\dot{x} = 2$ when $t = 0$, so that

$$0.3 = B,$$
$$2 = C\omega.$$

Hence $B = 0.3$ and $C = 2/\omega = 2/20 = 0.1$.

6. For simple harmonic motion,

$$x = B\cos\omega t + C\sin\omega t,$$

so

$$\ddot{x} = -B\omega^2\cos\omega t - C\omega^2\sin\omega t,$$

that is,

$$\ddot{x} = -\omega^2 x.$$

Differentiating this equation gives

$$\frac{d\ddot{x}}{dt} = -\omega^2\dot{x}.$$

But the acceleration $\ddot{x}$ has a maximum or minimum whenever $d\ddot{x}/dt = 0$, and the last equation shows that this occurs precisely when the velocity $\dot{x}$ is zero.

7. With the usual notation, we have

$$\begin{aligned} A &= \sqrt{B^2 + C^2} = \sqrt{0.3^2 + 0.1^2} \\ &= \sqrt{0.1} \simeq 0.316, \end{aligned}$$

and, as $C > 0$,

$$\begin{aligned} \phi &= -\arccos(B/A) = -\arccos(0.3/\sqrt{0.1}) \\ &\simeq -0.322. \end{aligned}$$

Thus $x \simeq 0.316\cos(20t - 0.322)$.

8. Since $x = l_0 + B\cos\omega t + C\sin\omega t$, we have

$$\dot{x} = -\omega B\sin\omega t + \omega C\cos\omega t.$$

Initially $x(0) = x_0$ and $\dot{x}(0) = v_0$, which leads to

$$x_0 = l_0 + B,$$
$$v_0 = \omega C.$$

Hence $B = x_0 - l_0$ and $C = v_0/\omega$, so the required solution is

$$x = l_0 + (x_0 - l_0)\cos\omega t + (v_0/\omega)\sin\omega t.$$

9.

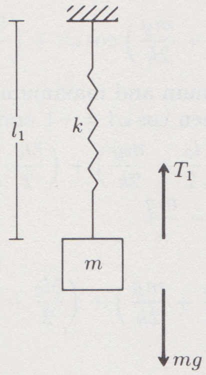

Figure 2

The extension of the spring is $l_1 - l_0$, and so the magnitude of the spring force is

$$T_1 = k(l_1 - l_0).$$

As the particle is in equilibrium,

$$mg = T_1 = k(l_1 - l_0).$$

Hence $l_1 = l_0 + mg/k$.

10. For the block to be in equilibrium the spring must be extended an amount

$$d = \frac{mg}{k} = \frac{3 \times 9.81}{250} \simeq 0.12\,\text{m}.$$

The greatest allowable value of d is

$$0.8 - l_0 = 0.8 - 0.5 = 0.3\,\text{m}.$$

The corresponding heaviest mass m is given by $0.3 = mg/k$, so that

$$m = 0.3k/g = 0.3 \times 250/9.81 \simeq 7.6\,\text{kg}.$$

11. The oscillation is given by

$$x(t) = A\cos(\omega t + \phi),$$

where $\omega = \sqrt{k/m}$. As seen earlier, this oscillation has period

$$\tau = 2\pi/\omega = 2\pi\sqrt{m/k}.$$

Also $d = mg/k$, so that $\tau = 2\pi\sqrt{d/g}$.

12.

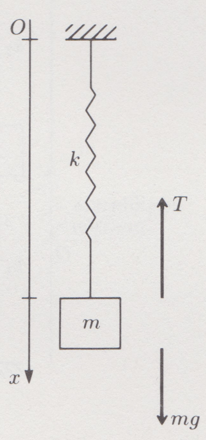

Figure 3

In order to derive the equation of motion, we assume that the spring is extended. The extension of the spring is $x - l_0$, and so the tension in the spring is

$$T = k(x - l_0).$$

Since the spring force acts upwards on the particle, the x-component of the total force acting on the particle is $mg - T$. Then Newton's second law gives

$$\begin{aligned} m\ddot{x} &= mg - T \\ &= mg - k(x - l_0), \end{aligned}$$

or $\quad m\ddot{x} + kx = mg + kl_0$.

Dividing through by m, we can rewrite this differential equation as

$$\ddot{x} + \omega^2 x = \omega^2\left(l_0 + \frac{mg}{k}\right),$$

where $\omega^2 = k/m$.

The differential equation is inhomogeneous. A particular solution is

$$x_p = l_0 + \frac{mg}{k}.$$

(Notice that this is just the equilibrium length of the spring.) The complementary function is

$$x_c = B\cos\omega t + C\sin\omega t,$$

so the general solution is

$$x = l_0 + \frac{mg}{k} + B\cos\omega t + C\sin\omega t.$$

Now the given initial conditions are $x(0) = l_0$ and $\dot{x}(0) = 0$. These lead to

$$l_0 = l_0 + \frac{mg}{k} + B,$$

$$0 = \omega C.$$

Hence $B = -mg/k$ and $C = 0$, so the required solution is

$$x = l_0 + \frac{mg}{k}(1 - \cos\omega t).$$

13.

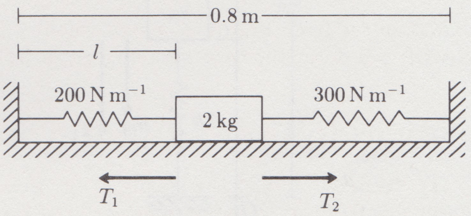

Figure 4

The sum of the natural lengths of the two springs is less than the distance between the two fixed surfaces; hence both springs will be in tension when the particle is in equilibrium. In this position, let the length of spring 1 be l. As the distance between the two walls is 0.8, the length of spring 2 in the equilibrium position is $0.8 - l$. The natural lengths of the two springs are respectively 0.3 and 0.4, so their extensions are $l - 0.3$ and $(0.8 - l) - 0.4 = 0.4 - l$. Hence the tensions in the two springs are

$$T_1 = 200(l - 0.3),$$

$$T_2 = 300(0.4 - l).$$

The particle can be in equilibrium only if these spring tensions are equal. So we have

$$200(l - 0.3) = 300(0.4 - l),$$

which leads to

$$l = 0.36.$$

In other words, the particle is in equilibrium at a point 0.36 m from the left-hand fixed surface (and 0.44 m from the right-hand surface).

14.

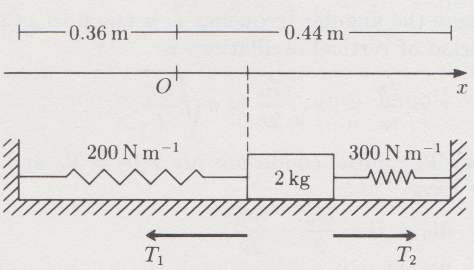

Figure 5

From Exercise 13, in equilibrium the lengths of the springs are respectively 0.36 and 0.44. If the particle is displaced to the right by a distance x from its equilibrium position then the lengths of the springs become $0.36 + x$ and $0.44 - x$. In this configuration, the extensions of the springs are

$$(0.36 + x) - 0.3 = 0.06 + x$$

and $$(0.44 - x) - 0.4 = 0.04 - x.$$

Hence the tensions in the two springs are

$$T_1 = 200(0.06 + x) = 12 + 200x,$$
$$T_2 = 300(0.04 - x) = 12 - 300x.$$

The equation of motion of the particle is

$$m\ddot{x} = T_2 - T_1,$$

that is,

$$2\ddot{x} = (12 - 300x) - (12 + 200x),$$

or $\ddot{x} + 250x = 0$, as required.

15. (i)

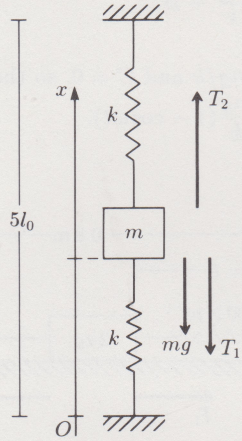

Figure 6

We assume that the two springs are both in tension. The lengths of the lower and upper springs are respectively x and $5l_0 - x$, so their extensions are $x - l_0$ and $4l_0 - x$. Hence the tensions in the lower and upper springs are

$$T_1 = k(x - l_0)$$

and $$T_2 = k(4l_0 - x).$$

The equation of motion is then

$$\begin{aligned} m\ddot{x} &= T_2 - T_1 - mg \\ &= k(4l_0 - x) - k(x - l_0) - mg, \end{aligned}$$

or $\quad m\ddot{x} + 2kx = 5kl_0 - mg.$

(ii) In equilibrium, with $x = x_e$ say, we have $\dot{x} = 0$ and $\ddot{x} = 0$. Substituting the second of these into the equation of motion gives

$$2kx_e = 5kl_0 - mg,$$

so the equilibrium height is

$$x_e = \frac{5l_0}{2} - \frac{mg}{2k}.$$

(iii) The general solution of the equation of motion is

$$x = B\cos\omega t + C\sin\omega t + x_e,$$

where the angular frequency ω is equal to $\sqrt{2k/m}$. The period of vertical oscillations is

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{2k}} = \pi\sqrt{\frac{2m}{k}}.$$

(iv) The initial conditions are $x(0) = 3l_0$ and $\dot{x}(0) = 0$. So we have

$$3l_0 = B + \frac{5l_0}{2} - \frac{mg}{2k},$$
$$0 = C\omega.$$

Hence $B = l_0/2 + mg/(2k)$ and $C = 0$, giving the particular solution

$$x(t) = \left(\frac{l_0}{2} + \frac{mg}{2k}\right)\cos\omega t + \left(\frac{5l_0}{2} - \frac{mg}{2k}\right).$$

(v) The minimum and maximum heights are attained respectively when $\cos\omega t = -1$ and when $\cos\omega t = 1$. So

$$\begin{aligned} x_{\text{min}} &= -\left(\frac{l_0}{2} + \frac{mg}{2k}\right) + \left(\frac{5l_0}{2} - \frac{mg}{2k}\right) \\ &= 2l_0 - \frac{mg}{k} \end{aligned}$$

and

$$\begin{aligned} x_{\text{max}} &= \left(\frac{l_0}{2} + \frac{mg}{2k}\right) + \left(\frac{5l_0}{2} - \frac{mg}{2k}\right) \\ &= 3l_0. \end{aligned}$$

16. Figure 7 shows the required force diagram for the equilibrium position, assuming both springs to be in tension. The spring force magnitudes are denoted by T_{01} and T_{02} respectively, and

$$T_{02} = T_{01} + mg. \tag{1}$$

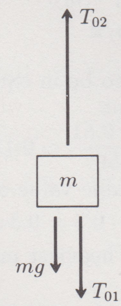

Figure 7

If the particle is moved upwards from its equilibrium position by an amount x, as shown in Figure 8, then the tension in spring 1 is increased by k_1x and the tension in spring 2 is decreased by k_2x.

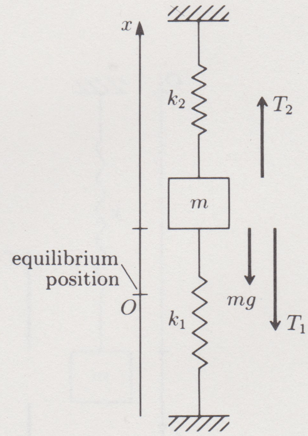

Figure 8

Hence the tension in spring 1 now has magnitude

$$T_1 = T_{01} + k_1x,$$

while that in spring 2 has magnitude

$$T_2 = T_{02} - k_2x.$$

From Figure 8, the x-component of the total force acting on the particle is $T_2 - T_1 - mg$, so Newton's second law gives

$$\begin{aligned} m\ddot{x} &= T_2 - T_1 - mg \\ &= T_{02} - k_2x - T_{01} - k_1x - mg. \end{aligned} \tag{2}$$

Now from Equation (1) we have $T_{02} - T_{01} = mg$; substituting this into Equation (2) gives

$$m\ddot{x} = mg - k_2 x - k_1 x - mg,$$

and hence $m\ddot{x} + (k_1 + k_2)x = 0$.

This is the required equation of motion and, as you might have expected, it is the same as Equation (14) of Section 2 for the horizontal two-spring system. (Note, however, that the equilibrium positions for these two systems are different.)

Solutions to the exercises in Section 3

1. **(i)** With the given x-axis, the force of gravity has x-component $F = -mg$. Using the definition of the potential energy function, we have

$$U(x) = -\int F(x)\,dx = \int mg\,dx = mgx + c.$$

The given datum is the origin; hence $U(0) = 0$ and $c = 0$. Thus

$$U(x) = mgx.$$

So for the force of gravity, the potential energy function is mg times the height above the chosen datum.

(ii) With the given x-axis, when the spring is extended, the magnitude of the spring force is kx and the force acts in the negative x-direction. So the spring force has x-component $F = -kx$. This expression is also valid when the spring is compressed.

Hence the potential energy function is

$$U(x) = -\int F(x)\,dx = \int kx\,dx = \tfrac{1}{2}kx^2 + c.$$

The given datum is $x = 0$; hence $U(0) = 0$ and $c = 0$. Thus

$$U(x) = \tfrac{1}{2}kx^2.$$

So for the spring force, the potential energy function is

$$\tfrac{1}{2} \times \text{ stiffness } \times \text{ (extension)}^2.$$

2. **(i)** For $F(x) = -ax^2$, we have

$$U(x) = -\int F(x)\,dx = \int ax^2\,dx = \tfrac{1}{3}ax^3 + c.$$

Choosing $x = 0$ as the datum, so that $U(0) = 0$, gives $c = 0$. Hence

$$U(x) = \tfrac{1}{3}ax^3.$$

(ii) In this case $F(x) = bx^{-2}$, so that

$$U(x) = -\int F(x)\,dx = -\int bx^{-2}\,dx = bx^{-1} + c.$$

Choosing $x = \infty$ as the datum, so that $U(\infty) = 0$, we have $c = 0$. Hence

$$U(x) = b/x.$$

(In each case, you may have chosen a different datum. However, your answers should differ from those above only by constants.)

3. **(i)** From the definition of potential energy, if $U(x)$ is the potential energy function corresponding to the force $F(x)$ then

$$\begin{aligned} U(x) &= -\int F(x)\,dx \\ &= -\int (F_1(x) + F_2(x))\,dx \\ &= -\int F_1(x)\,dx - \int F_2(x)\,dx \\ &= U_1(x) + U_2(x). \end{aligned}$$

(ii) If

$$F_1(x) = -ax^2 \quad \text{and} \quad F_2(x) = bx^{-2}$$

then, from Exercise 2, we have

$$U_1(x) = \tfrac{1}{3}ax^3 \quad \text{and} \quad U_2(x) = bx^{-1}.$$

Now $F(x) = F_1(x) + F_2(x)$, and so

$$\begin{aligned} U(x) &= U_1(x) + U_2(x) \\ &= \tfrac{1}{3}ax^3 + bx^{-1}. \end{aligned}$$

4. **(i)** Differentiating both sides of the equation

$$U(x) = -\int F(x)\,dx$$

and applying the definition of the indefinite integral gives

$$F(x) = -\frac{dU}{dx}.$$

(ii) If

$$U(x) = ax^2 + bx^{-2},$$

then

$$F(x) = -\frac{dU}{dx} = -2ax + 2bx^{-3}.$$

5. **(i)**

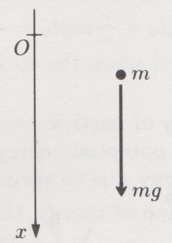

Figure 1

Using the origin as the datum, the gravitational potential energy is

$$\begin{aligned} U(x) &= mg \times \text{ height above datum} \\ &= -mgx. \end{aligned}$$

From the law of conservation of mechanical energy, we have

$$\tfrac{1}{2}mv^2 - mgx = E.$$

The initial condition is $v = 0$ when $x = 0$, and so $E = 0$. Hence

$$\tfrac{1}{2}v^2 - gx = 0,$$

or $\quad v^2 = 2gx$.

So when $x = 77.0$, we have

$$|v| = \sqrt{2 \times 9.81 \times 77.0} \simeq 38.9.$$

Hence the speed of the marble just before it hits the water is $38.9\,\mathrm{m\,s^{-1}}$.

(The same answer was obtained by direct application of Newton's second law in Example 1 and Exercise 7 of *Unit 4* Section 3.)

Further, when $v = 20$, we have

$$x = 20^2/(2 \times 9.81) \simeq 20.4.$$

So the marble has fallen 20.4 m when its speed reaches $20\,\mathrm{m\,s^{-1}}$.

(ii) The force of air resistance is a function of velocity. Hence the total force acting on the particle is not dependent on position alone, and the law of conservation of mechanical energy is not applicable.

6. The law of conservation of mechanical energy, $T + U = E$, gives

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = E.$$

Since $m = \frac{1}{2}$ and $k = 2$, this becomes

$$\tfrac{1}{4}v^2 + x^2 = E.$$

Initially $v = -3$ when $x = 2$, and so

$$E = \tfrac{9}{4} + 4 = \tfrac{25}{4}.$$

Hence

$$v^2 + 4x^2 = 25.$$

From this equation, $|v|$ will take its maximum value when $4x^2$ takes its minimum value, which is zero. So the maximum speed is $|v| = \sqrt{25} = 5$.

The particle is at rest when $v = 0$, which gives $x = \pm\frac{5}{2}$.

Summarizing, the maximum speed of the particle is $5\,\mathrm{m\,s^{-1}}$ and it is momentarily at rest at $x = \pm\frac{5}{2}$ m.

7. We choose A as the datum for gravitational potential energy. Initially the particle is at depth $\frac{5}{4}l_0$, the spring has extension $\frac{1}{4}l_0$, and the velocity of the particle is 0. So at the start of the motion we have

$$\begin{aligned}\text{kinetic energy of particle } &= 0,\\ \text{gravitational potential energy } &= -\tfrac{5}{4}mgl_0,\\ \text{potential energy due to spring } &= \tfrac{1}{2}(10mg/l_0)(\tfrac{1}{4}l_0)^2\\ &= \tfrac{5}{16}mgl_0.\end{aligned}$$

Using the result of Exercise 3(i), the particle's total potential energy is the sum of its gravitational and spring potential energies, so its total mechanical energy is

$$E = 0 - \tfrac{5}{4}mgl_0 + \tfrac{5}{16}mgl_0 = -\tfrac{15}{16}mgl_0.$$

We wish to find $|v|$ when the spring has its natural length, for which

$$\begin{aligned}\text{kinetic energy of particle } &= \tfrac{1}{2}mv^2,\\ \text{gravitational potential energy } &= -mgl_0,\\ \text{potential energy due to spring } &= 0.\end{aligned}$$

By the conservation of energy, therefore,

$$\tfrac{1}{2}mv^2 - mgl_0 + 0 = E = -\tfrac{15}{16}mgl_0,$$

which leads to

$$|v| = \sqrt{gl_0/8}.$$

8. **(i)** The potential energy function is

$$\begin{aligned}U(x) &= -\int F(x)\,dx\\ &= -\int (2 - 2x)\,dx\\ &= -2x + x^2,\end{aligned}$$

where we have chosen the origin as the datum (you may have made a different choice).

(ii) The total mechanical energy of the particle is

$$\begin{aligned}E &= \tfrac{1}{2}mv^2 + U(x)\\ &= v^2 - 2x + x^2,\end{aligned}$$

since the mass of the particle is $m = 2$. Now $v = 0$ when $x = -1$, and so

$$E = 0 + 2 + 1 = 3.$$

(If you used a different datum for the potential energy, so that $U(x) = -2x + x^2 + c$, then your answer for the total mechanical energy will be $E = 3 + c$.)

(iii) By the law of conservation of mechanical energy,

$$3 = v^2 - 2x + x^2,$$

so $v = \pm\sqrt{3 + 2x - x^2}$.

(iv) The region of motion is given by

$$3 + 2x - x^2 \geq 0,$$

or $(1 + x)(3 - x) \geq 0$,

that is,

$$-1 \leq x \leq 3.$$

At the mid-point, $x = 1$, the particle's speed is

$$|v| = \sqrt{3 + 2 - 1} = 2.$$

Solutions to the exercises in Section 4

1. The length of the spring is $(h^2 + x^2)^{1/2}$, and so its extension is $(h^2 + x^2)^{1/2} - l_0$. Hence

$$U(x) = \tfrac{1}{2}k[(h^2 + x^2)^{1/2} - l_0]^2,$$

and

$$\begin{aligned}U'(x) &= \frac{kx[(h^2 + x^2)^{1/2} - l_0]}{(h^2 + x^2)^{1/2}}\\ &= kx[1 - l_0(h^2 + x^2)^{-1/2}].\end{aligned}$$

The positions of equilibrium are given by $U'(x) = 0$. As $h > l_0$, the only position of equilibrium is at $x = 0$. Now

$$U''(x) = k[1 - l_0(h^2 + x^2)^{-1/2}] + kl_0x^2(h^2 + x^2)^{-3/2}.$$

Putting $x = 0$ here gives

$$U''(0) = k(1 - l_0/h),$$

so that the period of small oscillations about the position of equilibrium is

$$\tau = \frac{2\pi}{\omega} \simeq 2\pi\sqrt{\frac{m}{U''(0)}} = 2\pi\sqrt{\frac{mh}{k(h - l_0)}}.$$

2. The solution to this exercise is given in the television programme (see also Part 1 of the programme synopsis in Subsection 4.3).

3. **(i)** The potential energy of the car is

$$U(q) = mgh(q) = mgq^2/(4a),$$

so $U'(q) = mgq/(2a)$ and $U''(q) = mg/(2a)$.

The minimum of the potential energy function is at $q = 0$, and $U''(0) = mg/(2a)$. The period of small oscillations about the position of equilibrium is

$$\tau = 2\pi\sqrt{\frac{m}{U''(0)}} = 2\pi\sqrt{\frac{2a}{g}}.$$

(ii) The period of large oscillations is also $\tau = 2\pi\sqrt{2a/g}$, as the potential energy function is exactly quadratic in q (see the marginal note on page 30).

4. The total mechanical energy is

$$\begin{aligned}E &= \tfrac{1}{2}m\dot{q}^2 + U(q)\\ &= \tfrac{1}{2}m\dot{q}^2 + mgl\left(1 - \cos\frac{q}{l}\right).\end{aligned}$$

By the principle of conservation of energy, E is a constant. Differentiating this equation with respect to time therefore gives

$$m\dot{q}\ddot{q} + mg\dot{q}\sin\frac{q}{l} = 0.$$

Neglecting the possibility $\dot{q} = 0$ (which corresponds to the pendulum being stationary), we obtain the equation of motion

$$\ddot{q} + g\sin\frac{q}{l} = 0.$$

For small oscillations we have $\sin(q/l) \simeq q/l$, and the corresponding equation of motion is

$$\ddot{q} + \frac{g}{l}q \simeq 0.$$

This approximates the equation of simple harmonic motion with angular frequency $\omega = \sqrt{g/l}$. So the period of small oscillations is

$$\tau = \frac{2\pi}{\omega} \simeq 2\pi\sqrt{\frac{l}{g}}.$$

Solutions to the exercises in Section 5

1. (i) The extension of the spring is $x - 0.2$, and so its tension is

$$T = 50(x-0.2) = 50x - 10.$$

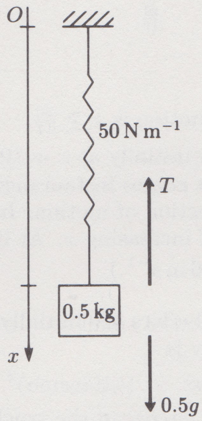

Figure 1

Hence the equation of motion of the particle is

$$0.5\ddot{x} = 0.5g - T = 5 - (50x - 10),$$

or $\ddot{x} + 100x = 30$.

(ii) In the equilibrium position we have $x =$ constant, and so $\dot{x} = 0$ and $\ddot{x} = 0$. Thus

$$100x = 30 \quad \text{or} \quad x = 0.3.$$

The length of the spring in the equilibrium position is therefore 0.3 m.

(iii) The general solution of the equation of motion is

$$x = 0.3 + B\cos 10t + C\sin 10t.$$

The initial conditions are $x(0) = 0.2$ and $\dot{x}(0) = 1$. Hence

$$0.2 = 0.3 + B$$

and $1 = 10C$,

which lead to $B = -0.1$ and $C = 0.1$. The spring length for this particular motion is given by

$$x = 0.3 - 0.1\cos 10t + 0.1\sin 10t.$$

(iv) The angular frequency of the simple harmonic motion is $\omega = \sqrt{50/0.5} = 10$. So the period is

$$\tau = 2\pi/\omega = \tfrac{1}{5}\pi \simeq 0.63\,\text{s}.$$

The amplitude of the oscillations is

$$A = \sqrt{B^2 + C^2} = \sqrt{0.02} \simeq 0.14\,\text{m}.$$

The phase of the oscillations is (as $C > 0$)

$$\phi = -\arccos(B/A) = -\arccos(-1/\sqrt{2}) = -3\pi/4.$$

(v) The first two derivatives of

$$x = 0.3 - 0.1\cos 10t + 0.1\sin 10t$$

are

$$\dot{x} = \sin 10t + \cos 10t$$

and $\ddot{x} = 10\cos 10t - 10\sin 10t$.

The particle is at its lowest point when the spring length x is a maximum, that is, when $\dot{x} = 0$ and $\ddot{x} < 0$. The condition $\dot{x} = 0$ gives

$$\sin 10t + \cos 10t = 0,$$

or $\tan 10t = -1$.

This is first satisfied for $10t = \frac{3}{4}\pi$ or $t = 3\pi/40 \simeq 0.24\,\text{s}$. Now $\ddot{x}(3\pi/40) = -20/\sqrt{2} < 0$, and so the particle *is* at its lowest point at this time. Thus the maximum length of the spring is

$$x = 0.3 + 0.2/\sqrt{2} \simeq 0.44\,\text{m}.$$

2. If the equilibrium length of the spring is l_1 then the corresponding extension is $l_1 - l_0$ and the tension is $T_1 = k(l_1 - l_0)$. For equilibrium we have $T_1 = mg$, and so

$$l_1 = l_0 + \frac{mg}{k}.$$

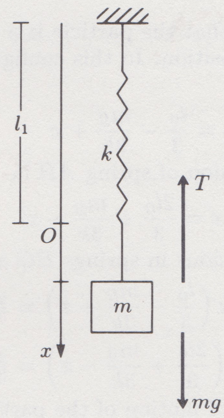

Figure 2

If the spring is now extended by a further amount x then the tension in the spring is

$$T = k(l_1 - l_0 + x) = mg + kx.$$

The equation of motion of the particle is

$$m\ddot{x} = mg - T,$$

or $m\ddot{x} + kx = 0$.

The general solution of this equation is

$$x = B\cos\omega t + C\sin\omega t,$$

where $\omega^2 = k/m$. Suppose that the particle is released at $t = 0$. Then the initial conditions are $x(0) = \frac{1}{4}l_0$ and $\dot{x}(0) = 0$, which give $B = \frac{1}{4}l_0$ and $C = 0$. Hence the required particular solution is

$$x = \tfrac{1}{4}l_0\cos\omega t.$$

3. First consider the equilibrium position. If the height of the particle B above C is l_1 then the length of spring BC is l_1, and the length of spring AB is $3l_0 - l_1$. So the extensions of the two springs are $l_1 - l_0$ (for BC) and $2l_0 - l_1$ (for AB), from which their tensions are

$$T_{10} = 2k(l_1 - l_0) \quad (\text{for } BC)$$

and $T_{20} = k(2l_0 - l_1)$ (for AB).

We are considering the equilibrium position, for which

$$T_{10} + mg = T_{20},$$

or $2k(l_1 - l_0) + mg = k(2l_0 - l_1)$.

Hence $l_1 = \frac{4}{3}l_0 - mg/(3k)$.

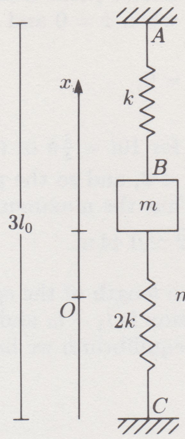

Figure 3

Now suppose that the particle is a height x above its equilibrium position. In this configuration, the extension of spring BC is

$$l_1 - l_0 + x = \frac{l_0}{3} - \frac{mg}{3k} + x$$

and the extension of spring AB is

$$2l_0 - l_1 - x = \frac{2l_0}{3} + \frac{mg}{3k} - x.$$

Hence the tensions in springs BC and AB are respectively

$$T_1 = 2k\left(\frac{l_0}{3} - \frac{mg}{3k} + x\right) = \tfrac{2}{3}kl_0 - \tfrac{2}{3}mg + 2kx$$

and $$T_2 = k\left(\frac{2l_0}{3} + \frac{mg}{3k} - x\right) = \tfrac{2}{3}kl_0 + \tfrac{1}{3}mg - kx.$$

The equation of motion of the particle is

$$\begin{aligned} m\ddot{x} &= T_2 - T_1 - mg \\ &= (\tfrac{2}{3}kl_0 + \tfrac{1}{3}mg - kx) - (\tfrac{2}{3}kl_0 - \tfrac{2}{3}mg + 2kx) - mg, \end{aligned}$$

or $m\ddot{x} + 3kx = 0$.

4. **(i)** The potential energy function is

$$U(x) = -\int F(x)\,dx = -\int \frac{2}{x^2}\,dx = \frac{2}{x},$$

where we have chosen $x = \infty$ as the datum. (If you chose a different datum then your answer should be $U(x) = 2x^{-1} + c$.)

(ii) The law of conservation of mechanical energy is

$$\tfrac{1}{2}mv^2 + U(x) = E,$$

or $\tfrac{3}{2}v^2 + \dfrac{2}{x} = E.$

Initially $v = -1$ when $x = 10$, which leads to

$$E = \tfrac{17}{10} \text{ joules.}$$

(If you used a different datum for the potential energy function then you should have found $E = \frac{17}{10} + c$.)

(iii) At the point of closest approach the particle must be stationary, so that $v = 0$. Then, from the law of conservation of mechanical energy, the corresponding value of x is given by

$$\tfrac{3}{2} \times 0^2 + \frac{2}{x} = \tfrac{17}{10}.$$

Hence the point of closest approach is $x = \frac{20}{17} \simeq 1.18\,\text{m}$.

(iv)

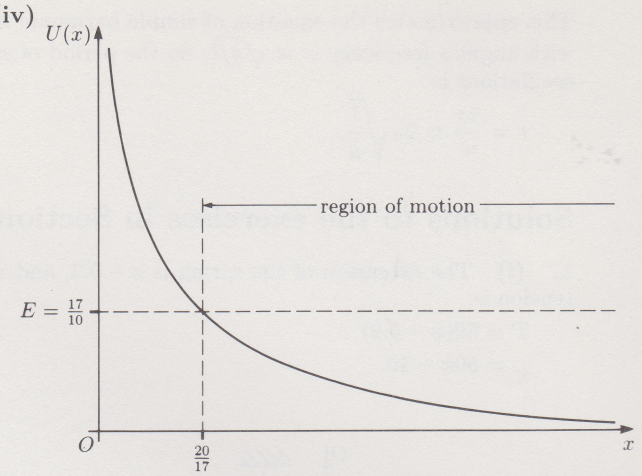

Figure 4

The region of motion is $x \geq \frac{20}{17}$.

(The particle is initially at $x = 10$, moving in the direction of decreasing x. It comes instantaneously to rest at $x = \frac{20}{17}$, and changes its direction of motion. It then continues to move in the direction of increasing x. At infinity its velocity will be $v = \sqrt{\frac{17}{15}} \simeq 1.06\,\text{m}\,\text{s}^{-1}$.)

5. When the buffers are initially compressed, the spring potential energy is

$$\tfrac{1}{2} \times \text{ stiffness } \times \text{ (extension)}^2 = \tfrac{1}{2} \times 10^5 \times (0.1)^2 = 500,$$

and the kinetic energy of the truck is zero. So the total mechanical energy of the system is 500 joules.

When the truck is on the point of leaving the buffers, the spring potential energy is zero and the kinetic energy of the truck is $\frac{1}{2} \times 2000v^2$, where v is the velocity of the truck. Using conservation of energy,

$$1000v^2 = 500$$

so $|v| = 1/\sqrt{2} = 0.707\,\text{m}\,\text{s}^{-1}$.

6. **(i)**

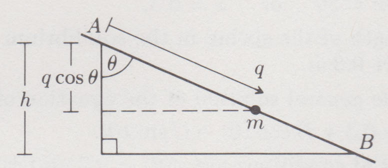

Figure 5

The height of the bead above the point B is $h - q\cos\theta$. The total mechanical energy of the bead is therefore

$$\begin{aligned} E &= \tfrac{1}{2} \times \text{ mass } \times \text{ (speed)}^2 + \text{ mass } \times g \times \text{ height} \\ &= \tfrac{1}{2}m\dot{q}^2 + mg(h - q\cos\theta). \end{aligned}$$

(ii) By conservation of energy we have $dE/dt = 0$, so that

$$\begin{aligned} 0 = \frac{dE}{dt} &= \tfrac{1}{2}m2\dot{q}\ddot{q} - mg\dot{q}\cos\theta \\ &= m\dot{q}(\ddot{q} - g\cos\theta). \end{aligned}$$

Thus when $\dot{q} \neq 0$ we obtain

$$\ddot{q} = g\cos\theta. \qquad (1)$$

(iii) Equation (1) has the general solution

$$q(t) = (\tfrac{1}{2}g\cos\theta)t^2 + Ct + D,$$

where C and D are arbitrary constants. Assuming the motion to start at $t = 0$, we have the initial conditions $q(0) = 0$ and $\dot{q}(0) = 0$, which give $C = D = 0$. The particular solution is therefore

$$q(t) = (\tfrac{1}{2}g\cos\theta)t^2.$$

Since $q = h/\cos\theta$ at B, the time $t = t_1$ at which the bead reaches B is given by

$$\frac{h}{\cos\theta} = (\tfrac{1}{2}g\cos\theta)t_1^2$$

or $$t_1 = \sqrt{\frac{2h}{g\cos^2\theta}} = \sqrt{\frac{2h}{g}}\sec\theta.$$

7. The potential energy function is

$$U(x) = -\int F(x)\,dx$$
$$= -\int (1-2x)\,dx = x^2 - x + c.$$

Taking $c = 0$ for convenience, the law of conservation of energy, $\frac{1}{2}mv^2 + U(x) = E$, gives

$$\tfrac{3}{2}v^2 + x^2 - x = E.$$

Initially $v = 2$ when $x = 0$, so $E = 6$ and

$$\tfrac{3}{2}v^2 + x^2 - x = 6.$$

The region of motion is given by

$$v^2 = \tfrac{2}{3}(6 - x^2 + x) \geq 0.$$

This condition is equivalent to

$$x^2 - x - 6 \leq 0,$$

which simplifies to

$$(x+2)(x-3) \leq 0$$

or $$-2 \leq x \leq 3.$$

8. (i) Suppose that the x-axis is chosen to point away from the wall, with its origin at the wall. Then the total force acting on the particle has x-component

$$F(x) = -\frac{A}{x^2} + \frac{B}{x^{10}}.$$

The particle can remain permanently at rest at $x = x_0$ if

$$F(x_0) = -\frac{A}{x_0^2} + \frac{B}{x_0^{10}} = 0,$$

which gives the value $x_0 = (B/A)^{1/8}$.

(ii) The potential energy function for the particle is

$$U(x) = -\int F(x)\,dx + c$$
$$= \int\left(\frac{A}{x^2} - \frac{B}{x^{10}}\right)dx + c$$
$$= -\frac{A}{x} + \frac{B}{9x^9} + c.$$

(The value of the arbitrary constant c will be chosen in part (iii) below.)

(iii) In order to find the period of small oscillations about x_0, we must approximate $U(x)$ by the first three terms of its Taylor series about the point $x = x_0$, giving

$$U(x) \simeq U(x_0) + (x - x_0)\left(\frac{dU}{dx}\right)_{x=x_0} + \tfrac{1}{2}(x-x_0)^2\left(\frac{d^2U}{dx^2}\right)_{x=x_0}.$$

Now $U(x) = -Ax^{-1} + \frac{1}{9}Bx^{-9} + c$, whose first two derivatives are

$$\frac{dU}{dx} = \frac{A}{x^2} - \frac{B}{x^{10}}$$

(which is, of course, just $-F(x)$) and

$$\frac{d^2U}{dx^2} = -\frac{2A}{x^3} + \frac{10B}{x^{11}}.$$

Putting $x = x_0$ into these expressions gives

$$U(x_0) = -\frac{A}{x_0} + \frac{B}{9x_0^9} + c,$$
$$\left(\frac{dU}{dx}\right)_{x=x_0} = \frac{A}{x_0^2} - \frac{B}{x_0^{10}} = -F(x_0) = 0$$

and $$\left(\frac{d^2U}{dx^2}\right)_{x=x_0} = -\frac{2A}{x_0^3} + \frac{10B}{x_0^{11}}.$$

The value of c can be chosen to be $Ax_0^{-1} - \frac{1}{9}Bx_0^{-9}$, so that $U(x_0) = 0$. Then

$$U(x) \simeq \tfrac{1}{2}k(x-x_0)^2,$$

where

$$k = \left(\frac{d^2U}{dx^2}\right)_{x=x_0} = -\frac{2A}{x_0^3} + \frac{10B}{x_0^{11}}$$
$$= \frac{2A}{x_0^3}\left(-1 + \frac{5B}{Ax_0^8}\right) = \frac{8A}{x_0^3},$$

since $x_0^8 = B/A$ from part (i). Hence the angular frequency of small oscillations is

$$\omega \simeq \sqrt{\frac{k}{m}} = \sqrt{\frac{8A}{mx_0^3}},$$

and the period of small oscillations is

$$\tau = \frac{2\pi}{\omega} \simeq \pi\sqrt{\frac{mx_0^3}{2A}}.$$

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